Differential Geometry and its Applications 29 (2011) 388–408

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

On real and complex Berwald connections associated to strongly convex weakly Kähler–Finsler metric

Chunping Zhong [1]

School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

ARTICLE INFO

Article history:
Received 5 August 2010
Received in revised form 16 December 2010
Available online 26 March 2011
Communicated by Z. Shen

MSC:
53C60
53C40

Keywords:
Complex Berwald metric
Complex Wrona metric

ABSTRACT

In this paper, we give a definition of weakly complex Berwald metric and prove that, (i) a strongly convex weakly Kähler–Finsler metric F on a complex manifold M is a weakly complex Berwald metric iff F is a real Berwald metric; (ii) assume that a strongly convex weakly Kähler–Finsler metric F is a weakly complex Berwald metric, then the associated real and complex Berwald connections coincide iff a suitable contraction of the curvature components of type $(2,0)$ of the complex Berwald connection vanish; (iii) the complex Wrona metric in $\mathbb{C}^n$ is a fundamental example of weakly complex Berwald metric whose holomorphic curvature and Ricci scalar curvature vanish identically. Moreover, the real geodesic of the complex Wrona metric on the Euclidean sphere $\mathbf{S}^{2n-1} \subset \mathbb{C}^n$ is explicitly obtained.

1. Introduction and statement of results

In real Finsler geometry, there are three canonical real Finsler connections associated to a given convex real Finsler metric, that is, the Cartan connection [1,14], the Chern–Rund connection [7,9,14], and the real Berwald connection [14]. All these Finsler connections are symmetric connections in the sense that their connection coefficients are symmetric in their lower indexes. They enjoy the same nonlinear connection coefficients, thus determine the same horizontal distribution [1,2]. It is known that the Cartan connection and the Chern–Rund connection enjoy the same horizontal connection coefficients, the vertical connection coefficients of the Chern–Rund connection and the real Berwald connection vanish identically. In general, however, the horizontal connection coefficients of the Cartan connection and the horizontal connection coefficients of the real Berwald connection are different [14]. A convex real Finsler metric is called a real Berwald metric if locally its associated horizontal real Berwald connection coefficients are independent of fiber coordinates [14]. It follows from Definition 25.1 and Proposition 25.1 in [14] that the horizontal real Berwald connection coefficients are independent of fiber coordinates iff the horizontal Cartan connection coefficients are independent of fiber coordinates. This implies that the definition of real Berwald metric is independent of the choice of the horizontal connection coefficients of the above canonical real Finsler connections that associated to a convex real Finsler metric.

In complex Finsler case, however, things are different. More precisely, in complex Finsler geometry, there are also three canonical complex Finsler connections associated to a given pseudoconvex complex Finsler metric on a complex manifold, that is, the Chern–Finsler connection [1,12,13], the complex Rund connection [15,17,20,21] and the complex Berwald connection [15,16]. All these complex Finsler connections are good complex vertical connection in the sense of [1]. The horizontal connection coefficients of the Chern–Finsler connection and complex Rund connection coincide. In general,

E-mail address: zcp@xmu.edu.cn.

[1] This work is supported by Program for New Century Excellent Talents in Fujian Province University, and the Natural Science Foundation of China (Grant Nos. 10971170, 10601040).

0926-2245/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.difgeo.2011.03.006

however, the horizontal connection coefficients of the Chern–Finsler connection and the horizontal connection coefficients of the complex Berwald connection are different, they coincide if and only if the given strongly pseudoconvex complex Finsler metric is a Kähler–Finsler metric. In [4,5], a strongly pseudoconvex complex Finsler metric is called a complex Berwald metric if the associated horizontal Chern–Finsler connection coefficients are independent of fiber coordinates. This definition was also used in [19] to give a characterization of complex Berwald metric with vanishing holomorphic curvature. The purpose of this paper is to give a definition of weakly complex Berwald metric via the independence of the fiber coordinates of the horizontal connection coefficients of the complex Berwald connection and give some characterizations of the weakly complex Berwald metric. It turns out that our definition of weakly complex Berwald metric is indeed weak than the definition of complex Berwald metric given in [4,5]. More precisely, we show that a complex Berwald metric is necessary a weakly complex Berwald metric but the converse, however, is not true. We shall give a counterexample, i.e., the complex Wrona metric in $\mathbb{C}^n$ [10], to confirm our assertion. Moreover, we obtain some fundamental theorems which relate real Berwald metric and weakly complex Berwald metric, as well as the real Berwald connection and complex Berwald connection, on strongly convex weakly Kähler–Finsler manifold and strongly convex Kähler–Finsler manifold, respectively.

The main results of this paper are

Theorem 1.1. *Let F be a strongly convex weakly Kähler–Finsler metric on a complex manifold M. Then F is a weakly complex Berwald metric if and only if F is a real Berwald metric.*

In general, there is hardly any fundamental relationship between a convex real Finsler metric and a strongly convex complex Finsler metric on a complex manifold. Theorem 1.1 shows that as far as the real Berwald metric and weakly complex Berwald metric are concerned, the condition of strongly convex weakly Kähler–Finsler metric plays an important role. In [18], the authors proved that if a strongly convex weakly Kähler–Finsler metric is a complex Berwald metric [4,5] then it is necessary a real Berwald metric. Theorem 1.1 is interesting in two aspects, first, our definition of weakly complex Berwald metric is indeed weak than the definition of complex Berwald metric given in [4,5]; second, the conclusion of Theorem 1.1 is necessary and sufficient. In Sections 5–6, we shall give a fundamental counterexample, i.e., the complex Wrona metric in $\mathbb{C}^n$ to show that our definition of weakly complex Berwald is weak than the definition of complex Berwald metric given in [4,5].

Since the complex Berwald connection and the complex Rund connection coincide on a Kähler–Finsler manifold, we immediately obtain the following characterization of complex Berwald metric, that is

Theorem 1.2. *Let F be a strongly convex Kähler–Finsler metric on a complex manifold M. Then F is a complex Berwald metric if and only if F is a real Berwald metric.*

Theorem 1.3. *Let F be a strongly convex weakly Kähler–Finsler metric on a complex manifold M. If F is a weakly complex Berwald metric with the curvature components*

$$\Phi^{\alpha}_{\beta;\mu\nu} = \mathscr{X}_{\mu}\big(\mathbb{G}^{\alpha}_{\beta\nu}\big) - \mathscr{X}_{\nu}\big(\mathbb{G}^{\alpha}_{\beta\mu}\big) + \mathbb{G}^{\gamma}_{\beta\nu}\mathbb{G}^{\alpha}_{\gamma\mu} - \mathbb{G}^{\gamma}_{\beta\mu}\mathbb{G}^{\alpha}_{\gamma\nu} \tag{1.1}$$

satisfying

$$\Phi^{\alpha}_{\beta;\mu\nu} v^{\beta} v^{\nu} = 0, \tag{1.2}$$

then the real and complex Berwald connections associated to F coincide.

Theorem 1.4. *Let F be a strongly convex Kähler–Finsler metric on a complex manifold M. If F is a complex Berwald metric, then the real and complex Berwald connections associated to F coincide.*

In Chapter 2 of [1], the authors gave a detailed comparison of the Cartan connection and the Chern–Finsler connection on strongly convex complex Finsler manifolds, and showed that even on a strongly convex Kähler–Finsler manifold, the Cartan connection and Chern–Finsler connection are differ, and their difference is not a tensor. The authors of [1] ask whether there exists another canonical connection in real Finsler manifold which agrees with the Chern–Finsler connection on Kähler–Finsler manifolds. The above Theorems 1.3 and 1.4 partially give an answer to this question.

Theorem 1.5. *The complex Wrona metric*

$$F(z,v) = \frac{\|v\|^2}{\sqrt{\|z\|^2\|v\|^2 - |\langle z,v\rangle|^2}}, \qquad \forall (z,v)\in\mathcal{D} \tag{1.3}$$

in $\mathbb{C}^n$ is a weakly complex Berwald metric with the holomorphic curvature and Ricci scalar curvature vanish identically, i.e.,

$$K(z,v)\equiv 0, \qquad Ric(z,v)\equiv 0. \tag{1.4}$$

Note that it makes no difference to define a real Berwald metric either using the horizontal connection coefficients of the Cartan connection or the real Berwald connection associated to a convex real Finsler metric. In [19], the author proved that a strongly convex Kähler–Finsler metric is a complex Berwald metric [4,5] with zero holomorphic curvature iff it is a complex locally Minkowski metric. The complex Wrona metric in $\mathbb{C}^n$, however, provides us a good example for weakly complex Berwald metric with vanishing holomorphic curvature, but obviously it is not a complex locally Minkowski metric, which in an aspect shows that our definition of weakly complex Berwald metric via the independence of fiber coordinates of the horizontal connection coefficients of the complex Berwald connection is indeed weak than the definition of complex Berwald metric [4,5].

Theorem 1.6. *Let F be the complex Wrona metric in $\mathbb{C}^n$ and $\sigma(t) = (\sigma^1(t), \ldots, \sigma^n(t))$ be a real geodesic of F. Then σ satisfies*

$$\left(\|\sigma\|^2\|\dot\sigma\|^2 - \left|\langle\sigma, \dot\sigma\rangle\right|^2\right)\ddot\sigma^\alpha = 2\|\dot\sigma\|^2\langle\sigma, \dot\sigma\rangle\dot\sigma^\alpha - \|\dot\sigma\|^4\sigma^\alpha, \quad \alpha = 1, \ldots, n. \tag{1.5}$$

For any given points $p, q \in \mathbf{S}^{2n-1}$ $(n \geqslant 2)$ with $\langle p, q\rangle = 0$, there exists a unique closed geodesic

$$\sigma(t) = \frac{1}{2}\left[\left(p - \sqrt{-1}q\right)e^{\sqrt{-1}t} + \left(p + \sqrt{-1}q\right)e^{-\sqrt{-1}t}\right], \quad t \in \mathbb{R} \tag{1.6}$$

on $\mathbf{S}^{2n-1}$ such that $\sigma(0) = p$, $\dot\sigma(0) = q$ and $\sigma, \dot\sigma \in \mathbf{S}^{2n-1}$ with $\langle\sigma, \dot\sigma\rangle = 0$. Moreover, the arc length $L(\sigma)$ of σ satisfies

$$L(\sigma) = 2\pi. \tag{1.7}$$

Note that the study of real (or complex) geodesic of the complex Wrona metric was posed as open problem in [10], where the author proved that if any real geodesic σ of the complex Wrona metric on $\mathbf{S}^{2n-1}$ $(n \geqslant 2)$ is parameterized by arc length t with $0 < t < \frac{\pi}{2}$, then the length $L(\sigma) = t$. Theorem 1.6 actually shows that among the real geodesics of the complex Wrona metric, there exists closed geodesic on $\mathbf{S}^{2n-1}$ $(n \geqslant 2)$ whose length is 2π. Thus Theorem 1.6 generalizes the proposition of real geodesic of the complex Wrona metric which was obtained in [10].

2. Notations

The material in this section is reproduced from Chapter 2 in [1] and is included here for the convenience of the reader as well as to establish notations.

Let M be a complex n-dimensional manifold with the canonical complex structure J. Denote by $T_{\mathbb{R}}M$ the real tangent bundle, and $T_{\mathbb{C}}M$ the complexified tangent bundle of M. Then J acts complex linearly on $T_{\mathbb{C}}M$ so that $T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$ is called the holomorphic tangent bundle of M. $T^{1,0}M$ is a complex manifold of complex dimension $2n$, and we also denote by J the complex structure on $T^{1,0}M$ if it causes no confusion.

Let $\{z^1, \ldots, z^n\}$ be a set of local complex coordinates on M, with $z^\alpha = x^\alpha + \sqrt{-1}x^{\alpha+n}$, so that $\{x^1, \ldots, x^n, x^{1+n}, \ldots, x^{2n}\}$ are local real coordinates on M. Denote by $\{z^1, \ldots, z^n, v^1, \ldots, v^n\}$ the induced complex coordinates on $T^{1,0}M$, with $v^\alpha = u^\alpha + \sqrt{-1}u^{\alpha+n}$, so that $\{x^1, \ldots, x^{2n}, u^1, \ldots, u^{2n}\}$ are local real coordinates on $T_{\mathbb{R}}M$.

In the following, lowercase Greek indices such as α, β, γ etc., will run from 1 to n, whereas lowercase roman indices such as a, b, c, etc., will run from 1 to $2n$, and the Einstein sum convention is assumed throughout the paper.

Set

$$\frac{\partial}{\partial z^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\frac{\partial}{\partial x^{\alpha+n}}\right), \qquad \frac{\partial}{\partial \overline{z^\alpha}} = \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + \sqrt{-1}\frac{\partial}{\partial x^{\alpha+n}}\right), \tag{2.1}$$

$$\frac{\partial}{\partial v^\alpha} = \frac{1}{2}\left(\frac{\partial}{\partial u^\alpha} - \sqrt{-1}\frac{\partial}{\partial u^{\alpha+n}}\right), \qquad \frac{\partial}{\partial \overline{v^\alpha}} = \frac{1}{2}\left(\frac{\partial}{\partial u^\alpha} + \sqrt{-1}\frac{\partial}{\partial u^{\alpha+n}}\right). \tag{2.2}$$

The bundles $T^{1,0}M$ and $T_{\mathbb{R}}M$ are isomorphic. We choose the explicit isomorphism ${}^o : T^{1,0}M \to T_{\mathbb{R}}M$ with its inverse ${}_o : T_{\mathbb{R}}M \to T^{1,0}M$, which are respectively given by

$$T_{\mathbb{R}}M \ni u = v^o = v + \bar v, \quad \forall v = v^\alpha\frac{\partial}{\partial z^\alpha} \in T^{1,0}M \tag{2.3}$$

and

$$T^{1,0}M \ni v = u_o = \frac{1}{2}(u - iJu), \quad \forall u = u^a\frac{\partial}{\partial x^a} \in T_{\mathbb{R}}M. \tag{2.4}$$

Both the bundle maps ^{o} and ${}_o$ are J preserving. In the following, $\tilde M$ denote either $T^{1,0}M$ or $T_{\mathbb{R}}M$ minus the zero section, depending on the actual situation. A local frame over $\mathbb{R}$ for $T_{\mathbb{R}}\tilde M$ is given by $\{\partial^o_1, \ldots, \partial^o_{2n}, \dot\partial^o_1, \ldots, \dot\partial^o_{2n}\}$, where

$$\partial^o_a = \frac{\partial}{\partial x^a} \quad \text{and} \quad \dot\partial^o_a = \frac{\partial}{\partial u^a}; \tag{2.5}$$

analogously, a local frame over $\mathbb{C}$ for $T^{1,0}\tilde{M}$ is given by $\{\partial_1,\ldots,\partial_n,\dot{\partial}_1,\ldots,\dot{\partial}_n\}$, where

$$\partial_\alpha = \frac{\partial}{\partial z^\alpha} \quad \text{and} \quad \dot{\partial}_\alpha = \frac{\partial}{\partial v^\alpha}. \tag{2.6}$$

We also denote by ^{o} and $_o$ the natural isomorphism $^o : T^{1,0}\tilde{M} \to T_{\mathbb{R}}\tilde{M}$ with its inverse $_o : T_{\mathbb{R}}\tilde{M} \to T^{1,0}\tilde{M}$. It is clear that

$$\partial^o_\alpha = (\partial_\alpha + \partial_{\overline{\alpha}}) = (\partial_\alpha)^o, \qquad \partial^o_{\alpha+n} = \sqrt{-1}(\partial_\alpha - \partial_{\overline{\alpha}}) = \left(\sqrt{-1}\partial_\alpha\right)^o, \tag{2.7}$$

$$\dot{\partial}^o_\alpha = (\dot{\partial}_\alpha + \dot{\partial}_{\overline{\alpha}}) = (\dot{\partial}_\alpha)^o, \qquad \dot{\partial}^o_{\alpha+n} = \sqrt{-1}(\dot{\partial}_\alpha - \dot{\partial}_{\overline{\alpha}}) = \left(\sqrt{-1}\dot{\partial}_\alpha\right)^o. \tag{2.8}$$

Definition 2.1. (See [1].) A real Finsler metric F on a manifold M is a function $F : T_{\mathbb{R}}M \to \mathbb{R}^+$ satisfying

(i) $G = F^2$ is smooth on $\tilde{M}$;
(ii) $F(p,u) > 0$ for $(p,u) \in \tilde{M}$;
(iii) $F(p,\lambda u) = |\lambda| F(p,u)$ for all $(p,u) \in T_{\mathbb{R}}M$ and $\lambda \in \mathbb{R}$.

If furthermore,

(iv) the Hessian matrix

$$(G_{ab}) = \left(\frac{\partial^2 G}{\partial u^a \partial u^b}\right) \tag{2.9}$$

is positive definite on $\tilde{M}$, then F is called a convex real Finsler metric.

Definition 2.2. (See [1].) A complex Finsler metric F on a complex manifold M is a continuous function $F : T^{1,0}M \to \mathbb{R}^+$ satisfying

(i) $G = F^2$ is smooth on $\tilde{M}$;
(ii) $F(p,v) > 0$ for all $(p,v) \in \tilde{M}$;
(iii) $F(p,\zeta v) = |\zeta| F(p,v)$ for all $(p,v) \in T^{1,0}M$ and $\zeta \in \mathbb{C}$.

If furthermore,

(iv) the Levi matrix (or complex Hessian matrix)

$$(G_{\alpha\overline{\beta}}) = \left(\frac{\partial^2 G}{\partial v^\alpha \partial \overline{v^\beta}}\right) \tag{2.10}$$

is positive definite on $\tilde{M}$, then F is called a strongly pseudoconvex complex Finsler metric.

Remark 2.3. In most cases, complex Finsler metrics are only smooth over a subset of $\tilde{M}$. For example, the complex Randers metrics [3] and the complex Wrona metric (see Section 5).

Let $F : T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex complex Finsler metric on a complex manifold M. Using the complex structure J on M and the bundle map $_o : T_{\mathbb{R}}M \to T^{1,0}M$ we can define a real function

$$F^o : T_{\mathbb{R}}M \to \mathbb{R}^+, \qquad F^o(u) := F(u_o), \quad \forall u \in T_{\mathbb{R}}M. \tag{2.11}$$

Definition 2.4. (See [1].) A strongly pseudoconvex complex Finsler metric F is called a strongly convex complex Finsler metric if the associated function F^o is a convex real Finsler metric.

If F is a strongly convex complex Finsler metric on a complex manifold M, then we use the same symbol F to denote the associated convex real Finsler metric F^o with the understanding that $F(u)$ is defined by $F(u_o)$ for $u \in T_{\mathbb{R}}M$.

In the following, we denote the derivatives of G with respect to the v-coordinates by

$$G_\alpha = \frac{\partial G}{\partial v^\alpha}, \qquad G_{\overline{\beta}} = \frac{\partial G}{\partial \overline{v^\beta}}, \qquad G_{\alpha\overline{\beta}} = \frac{\partial^2 G}{\partial v^\alpha \partial \overline{v^\beta}};$$

and the derivatives of G with respect to z-coordinates by indexes after a semicolon, i.e.,

$$G_{;\mu} = \frac{\partial G}{\partial z^\mu}, \qquad G_{\overline{\tau};\mu} = \frac{\partial^2 G}{\partial z^\mu \partial \overline{v^\tau}}.$$

If F is a strongly convex complex Finsler metric on a complex manifold M, we denote by $(G^{\overline{\beta}\alpha})$ and (G^{ba}) the inverse matrices of $(G_{\alpha\overline{\beta}})$ and (G_{ab}), respectively, such that $G_{\alpha\overline{\beta}}G^{\overline{\beta}\gamma} = \delta_\alpha^\gamma$ and $G_{ab}G^{bc} = \delta_a^c$.

Proposition 2.5. *Let F be a strongly convex complex Finsler metric on a complex manifold M. Then*

$$\det A = 2^{2n} \det B \cdot \det\bigl(B - C^* B^{-1} C\bigr), \tag{2.12}$$

where $A = (G_{ab})$, $B = (G_{\alpha\overline{\beta}})$, $C = (G_{\alpha\beta})$, and C^* is the Hermitian transpose of the matrix C.

Proof. Locally, one finds that [1, p. 113, Eq. (2.6.5)]

$$G_{ab} = \begin{cases} G_{\alpha\beta} + G_{\overline{\alpha}\beta} + G_{\alpha\overline{\beta}} + G_{\overline{\alpha}\overline{\beta}}, & \text{if } 1 \leqslant a, b \leqslant n, \\ \sqrt{-1}(G_{\alpha\beta} + G_{\overline{\alpha}\beta} - G_{\alpha\overline{\beta}} - G_{\overline{\alpha}\overline{\beta}}), & \text{if } 1 \leqslant a \leqslant n \text{ and } n+1 \leqslant b \leqslant 2n, \\ \sqrt{-1}(G_{\alpha\beta} - G_{\overline{\alpha}\beta} + G_{\alpha\overline{\beta}} - G_{\overline{\alpha}\overline{\beta}}), & \text{if } n+1 \leqslant a \leqslant 2n \text{ and } 1 \leqslant b \leqslant n, \\ -(G_{\alpha\beta} - G_{\overline{\alpha}\beta} - G_{\alpha\overline{\beta}} + G_{\overline{\alpha}\overline{\beta}}), & \text{if } n+1 \leqslant a, b \leqslant 2n. \end{cases}$$

Thus A is partitioned as

$$A = \begin{pmatrix} (G_{\alpha\beta} + G_{\overline{\alpha}\beta} + G_{\alpha\overline{\beta}} + G_{\overline{\alpha}\overline{\beta}}) & \sqrt{-1}(G_{\alpha\beta} + G_{\overline{\alpha}\beta} - G_{\alpha\overline{\beta}} - G_{\overline{\alpha}\overline{\beta}}) \\ \sqrt{-1}(G_{\alpha\beta} - G_{\overline{\alpha}\beta} + G_{\alpha\overline{\beta}} - G_{\overline{\alpha}\overline{\beta}}) & -(G_{\alpha\beta} - G_{\overline{\alpha}\beta} - G_{\alpha\overline{\beta}} + G_{\overline{\alpha}\overline{\beta}}) \end{pmatrix}.$$

Since the effect of a type 3 elementary operation, i.e., addition of a scalar multiple of one row (column) to another row (column), does not change the determinant. It follows that

$$\begin{aligned} \det A &= \begin{vmatrix} 2(G_{\overline{\alpha}\beta} + G_{\alpha\overline{\beta}}) & \sqrt{-1}(G_{\alpha\beta} - G_{\overline{\alpha}\overline{\beta}}) + \sqrt{-1}(G_{\overline{\alpha}\beta} - G_{\alpha\overline{\beta}}) \\ -2\sqrt{-1}(G_{\overline{\alpha}\overline{\beta}} - G_{\alpha\overline{\beta}}) & -(G_{\alpha\beta} + G_{\overline{\alpha}\overline{\beta}}) + (G_{\overline{\alpha}\beta} + G_{\alpha\overline{\beta}}) \end{vmatrix} \\ &= \begin{vmatrix} 4(G_{\alpha\overline{\beta}}) & 2\sqrt{-1}(G_{\alpha\beta} - G_{\alpha\overline{\beta}}) \\ -2\sqrt{-1}(G_{\overline{\alpha}\overline{\beta}} - G_{\alpha\overline{\beta}}) & -(G_{\alpha\beta} + G_{\overline{\alpha}\overline{\beta}}) + (G_{\overline{\alpha}\beta} + G_{\alpha\overline{\beta}}) \end{vmatrix} \\ &= \begin{vmatrix} 4(G_{\alpha\overline{\beta}}) & 2\sqrt{-1}(G_{\alpha\beta} - G_{\alpha\overline{\beta}}) \\ -2\sqrt{-1}(G_{\overline{\alpha}\overline{\beta}}) & -(G_{\overline{\alpha}\overline{\beta}} - G_{\overline{\alpha}\beta}) \end{vmatrix} \\ &= \begin{vmatrix} 4(G_{\alpha\overline{\beta}}) & 2\sqrt{-1}(G_{\alpha\beta}) \\ -2\sqrt{-1}(G_{\overline{\alpha}\overline{\beta}}) & (G_{\overline{\alpha}\beta}) \end{vmatrix} \\ &= 2^{2n}\begin{vmatrix} (G_{\alpha\overline{\beta}}) & \sqrt{-1}(G_{\alpha\beta}) \\ -\sqrt{-1}(G_{\overline{\alpha}\overline{\beta}}) & (G_{\overline{\alpha}\beta}) \end{vmatrix}, \end{aligned}$$

where the second equality is obtained by addition of $\sqrt{-1}$ times column 2 to column 1; the third equality is obtained by addition of $-\sqrt{-1}$ times row 2 to row 1; the fourth equality is obtained by addition of $-\frac{\sqrt{-1}}{2}$ times row 1 to row 2; the fifth equality is obtained by addition of $\frac{\sqrt{-1}}{2}$ times column 1 to column 2. If we denote by $C = (\sqrt{-1}G_{\alpha\beta})$, then it is a complex symmetric matrix so that $C^* = \overline{C}$. Thus we have

$$\det A = 2^{2n}\begin{vmatrix} B & C \\ C^* & B^* \end{vmatrix}. \tag{2.13}$$

If we denote by B^{-1} its inverse and let $D = -B^{-1}C$. Then we have

$$\begin{pmatrix} I & 0 \\ D^* & I \end{pmatrix}\begin{pmatrix} B & C \\ C^* & B^* \end{pmatrix}\begin{pmatrix} I & D \\ 0 & I \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & B^* - C^*B^{-1}C \end{pmatrix}, \tag{2.14}$$

where in the last equality we have use the equalities that $BD + C = \mathbf{0}$, $D^*B + C^* = \mathbf{0}$ and $D^*C + B^* = B^* - C^*B^{-1}C$. Note that B is a positive definite Hermitian symmetric matrix so that we have $B = B^*$, thus (2.12) follows immediately from (2.13) and (2.14). □

Remark 2.6. If $F = \sqrt{g_{i\overline{j}}(z)v^i\overline{v^j}}$ comes from a Hermitian metric on the complex manifold M, we have $C = (G_{\alpha\beta}) = \mathbf{0}$ and we obtain the following familiar formula

$$\sqrt{\det A} = 2^n \det B.$$

3. Connections on complex Finsler manifolds

If a complex manifold M is endowed with a strongly pseudoconvex complex Finsler metric F, then we call the pair (M,F) a strongly pseudoconvex complex Finsler manifold. It is well known that for a strongly pseudoconvex complex Finsler manifold (M,F), there exists three types of complex Finsler connections $D:\mathcal{X}(\mathcal{V}^{1,0})\to\mathcal{X}(T^*_{\mathbb{C}}\tilde{M}\otimes\mathcal{V}^{1,0})$, which are canonically associated to (M,F): the Chern–Finsler connection [1], the complex Rund connection [17] and the complex Berwald connection [15,16]. All these types of connections satisfy

$$D_X V=\left[dV^\beta(X)+V^\alpha\omega_\alpha^\beta(X)\right]\otimes\dot{\partial}_\beta,\qquad D_X\overline{V}=\overline{D_{\overline{X}}V}\tag{3.1}$$

for $V=V^\alpha\dot{\partial}_\alpha\in\mathcal{X}(\mathcal{V}^{1,0})$ and $X\in\mathcal{X}(T_{\mathbb{C}}\tilde{M})$, where ω_β^α in (3.1) denote the corresponding connection 1-forms of the Chern–Finsler connection, the complex Rund connection and the complex Berwald connection, respectively. More precisely, if we denote by

$$\Gamma^\alpha_{;\mu}=G^{\overline{\tau}\alpha}G_{\overline{\tau};\mu}\tag{3.2}$$

the canonical Chern–Finsler nonlinear connection coefficients that associated to F, and set

$$\delta_\mu=\partial_\mu-\Gamma^\alpha_{;\mu}\dot{\partial}_\alpha,\qquad \delta v^\mu=dv^\mu+\Gamma^\mu_{;\alpha}dz^\alpha,\tag{3.3}$$

such that $\{\delta_\mu,\dot{\partial}_\alpha\}$ is the local frame of $T^{1,0}\tilde{M}$ and $\{dz^\alpha,\delta v^\alpha\}$ is the local coframe dual to $\{\delta_\alpha,\dot{\partial}_\alpha\}$. Then the connection 1-forms ω^α_β of the Chern–Finsler connection that associated to F are characterized by

$$\omega^\alpha_\beta=\Gamma^\alpha_{\beta;\mu}dz^\mu+\Gamma^\alpha_{\beta\mu}\delta v^\mu,\tag{3.4}$$

where

$$\Gamma^\alpha_{\beta;\mu}=G^{\overline{\tau}\mu}\delta_\mu(G_{\beta\overline{\tau}}),\qquad \Gamma^\alpha_{\beta\mu}=G^{\overline{\tau}\mu}\dot{\partial}_\mu(G_{\beta\overline{\tau}})\tag{3.5}$$

are called the horizontal and vertical connection coefficients of the Chern–Finsler connection, respectively. It is a well-known fact that $\Gamma^\alpha_{\beta;\mu}$ and $\Gamma^\alpha_{;\mu}$ are related by

$$\Gamma^\alpha_{\beta;\mu}=\dot{\partial}_\beta\big(\Gamma^\alpha_{;\mu}\big),\qquad \Gamma^\alpha_{\beta;\mu}v^\beta=\Gamma^\alpha_{;\mu},\tag{3.6}$$

and the vertical connection coefficients $\Gamma^\alpha_{\beta\mu}$ satisfy

$$\Gamma^\alpha_{\beta\mu}v^\beta=0.\tag{3.7}$$

Moreover, since $\Gamma^\alpha_{\beta;\mu}$ is zero homogeneity with respect to v, it follows by Euler theorem for homogeneous functions that

$$\iota\big(\Gamma^\alpha_{\beta;\mu}\big)=0,\tag{3.8}$$

where $\iota=v^\gamma\dot{\partial}_\gamma$ is the complex radial vertical vector field on $\tilde{M}$.

The connection 1-form ω^α_β of the complex Rund connection that associated to F are characterized by

$$\omega^\alpha_\beta=\Gamma^\alpha_{\beta;\mu}dz^\mu,\tag{3.9}$$

where $\Gamma^\alpha_{\beta;\mu}$ are defined in (3.5).

It is clear from (3.9) that the connection 1-forms of the complex Rund connection are just the horizontal components of the connection 1-forms of the Chern–Finsler connection. Thus the horizontal covariant derivative of a vector field $X\in\mathcal{X}(\mathcal{V})$ with respect to the Chern–Finsler connection and the complex Rund connection are actually the same [1,17]. Moreover, the nonlinear connection coefficients associated respectively to the Chern–Finsler connection and the complex Rund connection are the same one. In the following we denote by $\mathcal{H}^{1,0}$ the complex horizontal bundle that associated to the Chern–Finsler connection (or the complex Rund connection since they are the same).

The main difference between the complex Berwald connection and the Chern–Finsler connection (or complex Rund connection) that associated to F is that their corresponding nonlinear connection coefficients are different. More precisely, if we set

$$2\mathbb{G}^\alpha:=\Gamma^\alpha_{;\mu}v^\mu,\qquad \mathbb{G}^\alpha_\mu:=\dot{\partial}_\mu\big(\mathbb{G}^\alpha\big),\tag{3.10}$$

where $\Gamma^\alpha_{;\mu}$ are given by (3.2). Then we have the following proposition.

Proposition 3.1. *Under local changing of coordinates on $T^{1,0}M$, the functions $\mathbb{G}^\alpha_\mu$ defined by (3.10) satisfy the transformation law of a complex nonlinear connection that defined on $T^{1,0}M$.*

Proof. In fact, if (U_A, z_A) and (U_B, z_B) are two complex coordinate neighborhoods on M such that $U_A \cap U_B \neq \emptyset$, and $\{\pi^{-1}(U_A), (z_A, v_A)\}$, $\{\pi^{-1}(U_B), (z_B, v_B)\}$ are the induced complex coordinate neighborhoods on $T^{1,0}M$. Then it is easy to check that

$$(\mathbb{G}_\alpha^\beta)_B = \frac{\partial z_B^\beta}{\partial z_A^\gamma}\frac{\partial z_A^\mu}{\partial z_B^\alpha}(\mathbb{G}_\mu^\gamma)_A - \frac{\partial^2 z_B^\beta}{\partial z_A^\gamma \partial z_A^\mu}\frac{\partial z_A^\gamma}{\partial z_B^\alpha} v_A^\mu.$$

This shows that $\mathbb{G}_\alpha^\beta$ satisfy the transformation law of a complex nonlinear connection that defined on $T^{1,0}M$. □

We call $\mathbb{G}_\mu^\alpha$ the nonlinear connection coefficients of the complex Berwald connection that associated to F, since by means of $\mathbb{G}_\mu^\alpha$ we can obtain a symmetric complex Finsler connection, that is the complex Berwald connection that associated to F. The connection 1-forms of the complex Berwald connection that associated to F are characterized by

$$\omega_\beta^\alpha = \mathbb{G}_{\beta\mu}^\alpha \, dz^\mu, \tag{3.11}$$

where

$$\mathbb{G}_{\beta\mu}^\alpha = \dot\partial_\beta(\mathbb{G}_\mu^\alpha) = \dot\partial_\beta\dot\partial_\mu(\mathbb{G}^\alpha) = \mathbb{G}_{\mu\beta}^\alpha. \tag{3.12}$$

Note that the nonlinear connection coefficients $\Gamma_{;\mu}^\alpha$ given by (3.2) satisfy

$$\Gamma_{;\mu}^\alpha(z, \lambda v) = \lambda \Gamma_{;\mu}^\alpha(z, v), \quad \forall \lambda \in \mathbb{C}^*,$$

where $\mathbb{C}^*$ denotes the set of all nonzero complex numbers. Thus by (3.10), we have

$$\mathbb{G}^\alpha(z, \lambda v) = \lambda^2 \mathbb{G}^\alpha(z, v), \qquad \mathbb{G}_\mu^\alpha(z, \lambda v) = \lambda \mathbb{G}_\mu^\alpha(z, v), \quad \forall \lambda \in \mathbb{C}^*. \tag{3.13}$$

Consequently by Euler theorem for homogeneous functions, we obtain

$$\mathbb{G}_{\beta\mu}^\alpha v^\beta = \mathbb{G}_\mu^\alpha \tag{3.14}$$

and we find that $\Gamma_{;\mu}^\alpha$ and $\mathbb{G}_\mu^\alpha$ are related by

$$2\mathbb{G}^\alpha = \mathbb{G}_\mu^\alpha v^\mu = \Gamma_{;\mu}^\alpha v^\mu. \tag{3.15}$$

It was proved in [1] that the Chern–Finsler connection associated to a strongly pseudoconvex complex metric F is a good complex vertical connection. We have

Proposition 3.2. *Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold M. Then both the complex Rund connection and the complex Berwald connection that associated to F are good complex vertical connections.*

Proof. First we notice that the connection 1-forms of the complex Rund connection and the complex Berwald connection are of type $(1,0)$. Define a bundle map $\Lambda : T^{1,0}\tilde M \to \mathcal{V}^{1,0}$ by

$$\Lambda(X) = D_X \iota, \quad \forall X \in T^{1,0}\tilde M, \tag{3.16}$$

where D denotes the complex Rund connection (or the complex Berwald connection). It is easy to check that restricting to the complex vertical bundle $\mathcal{V}^{1,0}$, Λ is an identity map. Thus the kernel of Λ defines a complex horizontal subbundle $\mathcal{H}^{1,0} \subset T^{1,0}\tilde M$, which implies that D is a good complex vertical connection (cf. Chapter 2 in [1]). □

Denote by $\mathscr{H}^{1,0}$ the complex horizontal subbundle that associated to the complex Berwald connection. Then $\{\mathscr{X}_1, \ldots, \mathscr{X}_n\}$ is a local frame for $\mathscr{H}^{1,0}$, here

$$\mathscr{X}_\alpha = \partial_\alpha - \mathbb{G}_\alpha^\beta \dot\partial_\beta. \tag{3.17}$$

Let $\Theta : \mathcal{V}^{1,0} \to \mathscr{H}^{1,0}$ denote the complex horizontal map associated to $\mathscr{H}^{1,0}$, Using Θ we can extend the Hermitian structure $\langle \cdot, \cdot \rangle$ on $\mathcal{V}^{1,0}$ to $\mathscr{H}^{1,0}$ via defining

$$\langle \mathscr{X}_\alpha, \mathscr{X}_\beta \rangle_{\mathbf{b}} =: \langle \Theta^{-1}(\mathscr{X}_\alpha), \Theta^{-1}(\mathscr{X}_\beta) \rangle_{\mathbf{b}} = \langle \dot\partial_\alpha, \dot\partial_\beta \rangle_{\mathbf{b}} = \langle \dot\partial_\alpha, \dot\partial_\beta \rangle = G_{\alpha\bar\beta}, \tag{3.18}$$

where $\langle \cdot, \cdot \rangle$ is the usual Hermitian inner product in $\mathcal{V}^{1,0}$ [1] and the above equality is evaluated at $v \in \tilde M$. Thus we can obtain a Hermitian structure $\langle \cdot, \cdot \rangle_{\mathbf{b}}$ in $T^{1,0}\tilde M$ by requiring $\mathscr{H}^{1,0}$ to be orthogonal to $\mathcal{V}^{1,0}$ and extend the connection on $\mathcal{V}^{1,0}$ to $\mathscr{H}^{1,0}$.

Proposition 3.3. *Let $\chi = v^\alpha \delta_\alpha$ and $\mathscr{X} = v^\alpha \mathscr{X}_\alpha$ be the complex horizontal radial vector field that associated to $\mathcal{H}^{1,0}$ and $\mathscr{H}^{1,0}$, respectively. Then*

$$\chi = \mathscr{X}. \tag{3.19}$$

Proof. This follows immediately from (3.15). □

In the following, we use the same symbol χ to denote the complex horizontal radial vector field that associated to either $\mathcal{H}^{1,0}$ or $\mathscr{H}^{1,0}$, since by Proposition 3.3 they are the same.

Definition 3.4. (See [1].) Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold M. F is called a strongly Kähler–Finsler metric iff $\Gamma^\alpha_{\mu;\beta} - \Gamma^\alpha_{\beta;\mu} = 0$; called a Kähler–Finsler metric iff $(\Gamma^\alpha_{\mu;\beta} - \Gamma^\alpha_{\beta;\mu})v^\mu = 0$; called a weakly Kähler–Finsler metric iff $G_\alpha(\Gamma^\alpha_{\mu;\beta} - \Gamma^\alpha_{\beta;\mu})v^\beta = 0$.

It was proved in [8] that F is a strongly Kähler–Finsler metric iff F is a Kähler–Finsler metric. So, in this paper we don't differentiate these two notions.

It is known that the Chern–Finsler connection is both horizontal and vertical metrical. For the complex Berwald connection, we have

Proposition 3.5. *Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold M. Then*

$$\mathscr{X}_\mu(G) = \mathscr{X}_{\overline{\mu}}(G) = 0 \tag{3.20}$$

if and only if F is a weakly Kähler–Finsler metric;

$$\mathscr{X}_{\overline{\mu}}(G_\alpha) = 0 \tag{3.21}$$

if and only if F is a Kähler–Finsler metric.

Proof. Using (3.6) and (3.10), we have

$$\mathbb{G}^\beta_\alpha = \frac{1}{2}\left(\Gamma^\beta_{\alpha;\mu} + \Gamma^\beta_{\mu;\alpha}\right)v^\mu, \tag{3.22}$$

from which we get

$$\mathscr{X}_\mu(G) = \delta_\mu(G) - \frac{1}{2}\left(\Gamma^\alpha_{\mu;\beta} - \Gamma^\alpha_{\beta;\mu}\right)v^\beta G_\alpha, \tag{3.23}$$

$$\mathscr{X}_{\overline{\mu}}(G_\alpha) = \delta_{\overline{\mu}}(G_\alpha) - \frac{1}{2}\left(\overline{\Gamma^\gamma_{\mu;\beta}} - \overline{\Gamma^\gamma_{\beta;\mu}}\right)\overline{v^\beta} G_{\alpha\overline{\gamma}}. \tag{3.24}$$

Since by Lemma 2.3.4 in [1],

$$\delta_\mu(G) = \delta_{\overline{\mu}}(G) = 0, \qquad \delta_{\overline{\mu}}(G_\alpha) = 0.$$

Thus $\mathscr{X}_\mu(G) = 0$ if and only if $G_\alpha(\Gamma^\alpha_{\mu;\beta} - \Gamma^\alpha_{\beta;\mu})v^\beta = 0$, i.e., F is a weakly Kähler–Finsler metric on M; $\mathscr{X}_{\overline{\mu}}(G_\alpha) = 0$ if and only if $(\Gamma^\gamma_{\mu;\beta} - \Gamma^\gamma_{\beta;\mu})v^\beta = 0$, i.e., F is a Kähler–Finsler metric on M. □

Proposition 3.6. *Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold M. Then the complex Berwald connection that associated to F is horizontal metrical if and only if F is a Kähler–Finsler metric.*

Proof. If the complex Berwald connection is horizontal metrical, then

$$G_{\beta\overline{\tau}|\mu} := \mathscr{X}_\mu(G_{\beta\overline{\tau}}) - G_{\alpha\overline{\tau}}\mathbb{G}^\alpha_{\beta;\mu} = 0,$$

i.e.,

$$\mathbb{G}^\alpha_{\beta;\mu} = G^{\overline{\tau}\alpha}\mathscr{X}_\mu(G_{\beta\overline{\tau}}) = \Gamma^\alpha_{\beta;\mu} - \frac{1}{2}\left(\Gamma^\sigma_{\mu;\nu} - \Gamma^\sigma_{\nu;\mu}\right)v^\nu \Gamma^\alpha_{\beta\sigma}, \tag{3.25}$$

where $\Gamma^\alpha_{\beta;\mu}$ and $\Gamma^\alpha_{\beta\mu}$ are given by (3.5). Thus in view of (3.22) we obtain

$$\frac{1}{2}\left[\dot{\partial}_\beta\left(\Gamma^\alpha_{\mu;\gamma}\right)v^\gamma + \Gamma^\alpha_{\mu;\beta} + \Gamma^\alpha_{\beta;\mu}\right] = \Gamma^\alpha_{\beta;\mu} - \frac{1}{2}\left(\Gamma^\sigma_{\mu;\nu} - \Gamma^\sigma_{\nu;\mu}\right)v^\nu \Gamma^\alpha_{\beta\sigma}. \tag{3.26}$$

Contracting (3.26) with v^β and using the identities (3.7) and (3.8), we get

$$\left(\Gamma^\alpha_{\beta;\mu} - \Gamma^\alpha_{\mu;\beta}\right)v^\beta = 0,$$

hence F is necessary a Kähler–Finsler metric. Conversely, if F is a Kähler–Finsler metric, then $\mathscr{X}_\mu = \delta_\mu$ and consequently the complex Berwald connection coincides with the complex Rund connection that associated to F, hence it is horizontal metrical. $\square$

4. Relationship between real and complex Berwald metrics

In real Finsler geometry, the horizontal bundles associated respectively to the Cartan connection, the Chern–Rund connection and the real Berwald connection are exactly the same one [2]. This is because these three connections enjoy the same nonlinear connection coefficients $\hat{\Gamma}^b_a$ that associated to a given real convex Finsler metric F, see [14].

Let F be a strongly pseudoconvex complex Finsler metric on M which is also strongly convex. Then it makes sense to talk about the real Berwald connection, as well as the complex Berwald connection that associated to F. Denote by $\hat{\Gamma}^b_{c;a}(x,u)$ the horizontal connection coefficients of the real Berwald connection that associated to F, then [14]

$$\hat{\Gamma}^b_{c;a}(x,u) = \dot{\partial}^o_c\left(\hat{\Gamma}^b_a\right), \tag{4.1}$$

where $\hat{\Gamma}^b_a$ are the nonlinear connection coefficients of the Cartan connection associated to F. As well known [14],

$$\hat{\Gamma}^b_{c;a}u^c = \hat{\Gamma}^b_a = \check{\Gamma}^b_{c;a}u^b,$$

where $\check{\Gamma}^b_{c;a}$ are the horizontal connection coefficients of the Cartan connection that associated to F. In general, however,

$$\hat{\Gamma}^b_{c;a} \neq \check{\Gamma}^b_{c;a}.$$

Definition 4.1. (See [14].) Let $\hat{\Gamma}^b_{c;a}(x,u)$ be the horizontal connection coefficients of the real Berwald connection that associated to a convex real Finsler metric F. If locally $\hat{\Gamma}^b_{c;a}$ are independent of the fiber coordinates u, then F is called a real Berwald metric.

It follows immediately from Definition 25.1 and Proposition 25.1 in [14] that

Proposition 4.2. *(See [14].) Let $\hat{\Gamma}^b_{c;a}$ and $\check{\Gamma}^b_{c;a}$ be respectively the horizontal connection coefficients of the real Berwald connection and the Cartan connection that associated to a convex real Finsler metric F. Then $\hat{\Gamma}^b_{c;a}$ are independent of the fiber coordinates u if and only if $\check{\Gamma}^b_{c;a}$ are independent of u.*

Definition 4.3. (See [4,5].) Let F be a strongly pseudoconvex complex Finsler metric on M. If locally the horizontal connection coefficients $\Gamma^\alpha_{\beta;\mu}$ of the associated Chern–Finsler connection are independent of the fibre coordinates v, then F is called a complex Berwald metric.

Remark 4.4. In [6], a complex Finsler manifold (M,F) satisfies Definition 4.3 is also called a complex manifold modeled on a complex Minkowski space. It is clear that any Hermitian metric and locally complex Minkowski metric belong to complex Berwald metrics.

In this paper, we give the following definition of weakly complex Berwald metric. It turns out that our definition of weakly complex Berwald metric is indeed weak than the definition of complex Berwald metric [4,5].

Definition 4.5. Let F be a strongly pseudoconvex complex Finsler metric on M. If locally the horizontal connection coefficients $\mathbb{G}^\alpha_{\beta\mu}$ of the associated complex Berwald connection are independent of the fibre coordinates v, then F is called a weakly complex Berwald metric.

Remark 4.6. It is easy to check that Definition 4.5 is independent of the choice of local holomorphic coordinates on M.

By Proposition 4.2, it makes no difference to define a real Berwald metric via the independence of fiber coordinates of either the horizontal real Berwald connection coefficients $\hat{\Gamma}^c_{a;b}$ or the horizontal Cartan connection coefficients $\check{\Gamma}^c_{a;b}$. In complex Finsler case, one may wonder whether our Definition 4.5 is indeed weak than Definition 4.3. We have

Theorem 4.7. *Let F be a strongly pseudoconvex complex Finsler metric on a complex manifold M. If F is a complex Berwald metric, then it is necessary a weakly complex Berwald metric, but the converse is not true.*

Proof. In fact, if F is a complex Berwald metric, then it follows from (3.6), (3.12) and (3.15) that

$$2\mathbb{G}^{\alpha}_{\beta\mu}=\dot\partial_\beta\dot\partial_\mu\big(\mathbb{G}^\alpha_\nu v^\nu\big)=\dot\partial_\beta\dot\partial_\mu\big(\Gamma^\alpha_{;\nu}v^\nu\big)=\dot\partial_\beta\dot\partial_\mu\big[\Gamma^\alpha_{\gamma;\nu}(z)v^\gamma v^\nu\big]=\Gamma^\alpha_{\mu;\beta}(z)+\Gamma^\alpha_{\beta;\mu}(z),$$

this implies that $\mathbb{G}^\alpha_{\beta\mu}$ are independent of fiber coordinates v, thus F is a weakly complex Berwald metric.

The assertion of the converse will be confirmed in Section 6 by showing that the complex Wrona metric in $\mathbb{C}^n$ is a weakly complex Berwald metric but not a complex Berwald metric. □

It is well known that a strongly pseudoconvex complex Finsler metric and a convex real Finsler metric is not so tightly related as that of the Hermitian and Riemannian metrics. Let F be a strongly convex complex Finsler metric on a complex manifold M. Then one may ask whether there is relationship between real Berwald metric and weakly complex Berwald metric. Now we shall answer this question. First we need a proposition (cf. Proposition 2.6.2 in [1]).

Let χ be the complex horizontal radial vector field that associated to the nonlinear connection coefficients $\mathbb{G}^\alpha_\mu$ of the complex Berwald connection, i.e.,

$$\chi(v)=v^\alpha\big(\partial_\alpha-\mathbb{G}^\beta_\alpha\dot\partial_\beta\big)=v^\alpha\big(\partial_\alpha-\Gamma^\beta_{;\alpha}\dot\partial_\beta\big),\quad \forall v\in T^{1,0}M. \tag{4.2}$$

Using χ and the bundle isomorphisms $_o: T_{\mathbb{R}}M\to T^{1,0}M$ and $^o: T^{1,0}M\to T_{\mathbb{R}}M$, we can define a section χ^o of $T_{\mathbb{R}}\tilde M$ by setting

$$\chi^o(u)=\big(\chi(u_o)\big)^o,\quad \forall u\in T_{\mathbb{R}}M. \tag{4.3}$$

Now if F is a strongly convex complex Finsler metric on the complex manifold M, then F is also a convex real Finsler metric on M, so that there is a real horizontal radial vector field $\hat\chi\in\mathcal{X}(\mathcal{H}_{\mathbb{R}})$, i.e.,

$$\hat\chi(u)=u^a\hat\delta_a=u^a\big(\partial^o_a-\hat\Gamma^b_a\dot\partial^o_b\big), \tag{4.4}$$

where $\hat\Gamma^b_a$ are the nonlinear connection coefficients of the Cartan connection associated to the convex real Finsler metric F.

Proposition 4.8. *(See [1].) Let F be a strongly convex weakly Kähler–Finsler metric on a complex manifold M. Then*

$$\chi=\hat\chi. \tag{4.5}$$

Now we are in a position to prove the following theorem.

Theorem 4.9. *Let F be a strongly convex weakly Kähler–Finsler metric on a complex manifold M. Then F is a weakly complex Berwald metric if and only if F is a real Berwald metric.*

Proof. It is clear that

$$\chi^o=u^a\partial^o_a-\frac{1}{2}\big(v^\alpha\Gamma^\beta_{;\alpha}+\overline{v^\alpha}\,\overline{\Gamma^\beta_{;\alpha}}\big)\dot\partial^o_\beta-\frac{1}{2\sqrt{-1}}\big(v^\alpha\Gamma^\beta_{;\alpha}-\overline{v^\alpha}\,\overline{\Gamma^\beta_{;\alpha}}\big)\dot\partial^o_{\beta+n},$$

or equivalently,

$$\chi^o=u^a\partial^o_a-\big(\mathbb{G}^\beta+\overline{\mathbb{G}^\beta}\big)\dot\partial^o_\beta+\sqrt{-1}\big(\mathbb{G}^\beta-\overline{\mathbb{G}^\beta}\big)\dot\partial^o_{\beta+n}. \tag{4.6}$$

It follows from (4.4) and Proposition 4.8 that

$$u^a\hat\Gamma^\beta_a=\mathbb{G}^\beta+\overline{\mathbb{G}^\beta}, \tag{4.7}$$

$$u^a\hat\Gamma^{\beta+n}_a=-\sqrt{-1}\big(\mathbb{G}^\beta-\overline{\mathbb{G}^\beta}\big). \tag{4.8}$$

Since the real Berwald connection coefficients $\hat\Gamma^b_{c;a}$ of F satisfy

$$\hat\Gamma^b_{c;a}=\hat\Gamma^b_{a;c},\qquad u^c\hat\Gamma^b_{c;a}=\hat\Gamma^b_a,$$

from which we get $\dot\partial^o_c(u^a\hat\Gamma^b_a)=2\hat\Gamma^b_c$. Thus in view of (4.7) we have

$$\hat\Gamma^\beta_\gamma=\frac{1}{2}\dot\partial^o_\gamma\big(u^a\hat\Gamma^\beta_a\big)=\mathrm{Re}\big[\big(\mathbb{G}^\beta_\gamma+\dot\partial_\gamma\big(\overline{\mathbb{G}^\beta}\big)\big]. \tag{4.9}$$

Similar calculations give

$$\hat{\Gamma}^{\beta}_{\gamma+n} = \frac{1}{2}\dot{\partial}^{o}_{\gamma+n}\big(u^a\hat{\Gamma}^{\beta}_{a}\big) = -\operatorname{Im}\big[\mathbb{G}^{\beta}_{\gamma} + \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big], \tag{4.10}$$

$$\hat{\Gamma}^{\beta+n}_{\gamma} = \frac{1}{2}\dot{\partial}^{o}_{\gamma}\big(u^a\hat{\Gamma}^{\beta+n}_{a}\big) = \operatorname{Im}\big[\mathbb{G}^{\beta}_{\gamma} - \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big], \tag{4.11}$$

$$\hat{\Gamma}^{\beta+n}_{\gamma+n} = \frac{1}{2}\dot{\partial}^{o}_{\gamma+n}\big(u^a\hat{\Gamma}^{\beta+n}_{a}\big) = \operatorname{Re}\big[\mathbb{G}^{\beta}_{\gamma} - \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big]. \tag{4.12}$$

Applying operator $\dot{\partial}^{o}_{\alpha} = \dot{\partial}_{\alpha} + \dot{\partial}_{\overline{\alpha}}$ to (4.9) and in view of (4.1), we obtain

$$\hat{\Gamma}^{\beta}_{\alpha;\gamma} = \operatorname{Re}\big[\mathbb{G}^{\beta}_{\alpha\gamma} + \dot{\partial}_{\alpha}\big(\overline{\mathbb{G}^{\beta}_{\gamma}}\big) + \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}_{\alpha}}\big) + \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big]. \tag{4.13}$$

Applying operator $\dot{\partial}^{o}_{\alpha+n} = \sqrt{-1}(\dot{\partial}_{\alpha} - \dot{\partial}_{\overline{\alpha}})$ to (4.9) and in view of (4.1), we obtain

$$\hat{\Gamma}^{\beta}_{\alpha+n;\gamma} = -\operatorname{Im}\big[\mathbb{G}^{\beta}_{\alpha\gamma} + \dot{\partial}_{\alpha}\big(\overline{\mathbb{G}^{\beta}_{\gamma}}\big) - \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}_{\alpha}}\big) + \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big]. \tag{4.14}$$

By similar calculations, we obtain

$$\hat{\Gamma}^{\beta}_{\alpha;\gamma+n} = -\operatorname{Im}\big[\mathbb{G}^{\beta}_{\alpha\gamma} - \dot{\partial}_{\alpha}\big(\overline{\mathbb{G}^{\beta}_{\gamma}}\big) + \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}_{\alpha}}\big) + \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big], \tag{4.15}$$

$$\hat{\Gamma}^{\beta}_{\alpha+n;\gamma+n} = -\operatorname{Re}\big[\mathbb{G}^{\beta}_{\alpha\gamma} - \dot{\partial}_{\alpha}\big(\overline{\mathbb{G}^{\beta}_{\gamma}}\big) - \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}_{\alpha}}\big) + \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big], \tag{4.16}$$

$$\hat{\Gamma}^{\beta+n}_{\alpha;\gamma} = \operatorname{Im}\big[\mathbb{G}^{\beta}_{\alpha\gamma} - \dot{\partial}_{\alpha}\big(\overline{\mathbb{G}^{\beta}_{\gamma}}\big) - \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}_{\alpha}}\big) - \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big], \tag{4.17}$$

$$\hat{\Gamma}^{\beta+n}_{\alpha+n;\gamma} = \operatorname{Re}\big[G^{\beta}_{\alpha\gamma} - \dot{\partial}_{\alpha}\big(\overline{\mathbb{G}^{\beta}_{\gamma}}\big) + \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}_{\alpha}}\big) - \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big], \tag{4.18}$$

$$\hat{\Gamma}^{\beta+n}_{\alpha;\gamma+n} = \operatorname{Re}\big[\mathbb{G}^{\beta}_{\alpha\gamma} + \dot{\partial}_{\alpha}\big(\overline{\mathbb{G}^{\beta}_{\gamma}}\big) - \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}_{\alpha}}\big) - \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big], \tag{4.19}$$

$$\hat{\Gamma}^{\beta+n}_{\alpha+n;\gamma+n} = -\operatorname{Im}\big[\mathbb{G}^{\beta}_{\alpha\gamma} + \dot{\partial}_{\alpha}\big(\overline{\mathbb{G}^{\beta}_{\gamma}}\big) + \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}_{\alpha}}\big) - \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big]. \tag{4.20}$$

Now assume that F is a weakly complex Berwald metric on M, then by Definition 4.5, $\mathbb{G}^{\alpha}_{\mu\gamma} = \mathbb{G}^{\alpha}_{\mu\gamma}(z)$ are independent of fiber coordinates v. Thus $\mathbb{G}^{\beta}$, $\mathbb{G}^{\beta}_{\gamma}$ are holomorphic with respect to the fiber coordinates v because

$$\mathbb{G}^{\beta} = \frac{1}{2}\mathbb{G}^{\beta}_{\gamma}v^{\gamma}, \qquad \mathbb{G}^{\beta}_{\gamma} = v^{\mu}\mathbb{G}^{\beta}_{\mu\gamma}(z).$$

Therefore,

$$\dot{\partial}_{\alpha}\big(\overline{\mathbb{G}^{\beta}_{\gamma}}\big) = \dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}_{\alpha}}\big) = \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big) = 0. \tag{4.21}$$

Substituting (4.21) into (4.13)–(4.20), we see that $\hat{\Gamma}^{b}_{a;c}$ are independent of the fiber coordinates u, so that F is a real Berwald metric on M.

Conversely, if F is a real Berwald metric on M. Then by (4.13) and (4.16), we get

$$\hat{\Gamma}^{\beta}_{\alpha;\gamma} - \hat{\Gamma}^{\beta}_{\alpha+n;\gamma+n} = 2\operatorname{Re}\big[\mathbb{G}^{\beta}_{\alpha\gamma} + \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big]. \tag{4.22}$$

By (4.18) and (4.19), we get

$$\hat{\Gamma}^{\beta+n}_{\alpha+n;\gamma} + \hat{\Gamma}^{\beta+n}_{\alpha;\gamma+n} = 2\operatorname{Re}\big[\mathbb{G}^{\beta}_{\alpha\gamma} - \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big]. \tag{4.23}$$

It follows from (4.22) and (4.23) that

$$\operatorname{Re}\big\{\mathbb{G}^{\beta}_{\alpha\gamma}\big\} = \frac{1}{4}\big[\hat{\Gamma}^{\beta}_{\alpha;\gamma} - \hat{\Gamma}^{\beta}_{\alpha+n;\gamma+n} + \hat{\Gamma}^{\beta+n}_{\alpha+n;\gamma} + \hat{\Gamma}^{\beta+n}_{\alpha;\gamma+n}\big]. \tag{4.24}$$

Similarly, by (4.14) and (4.15), we have

$$\hat{\Gamma}^{\beta}_{\alpha+n;\gamma} + \hat{\Gamma}^{\beta}_{\alpha;\gamma+n} = -2\operatorname{Im}\big[\mathbb{G}^{\beta}_{\alpha\gamma} + \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big]. \tag{4.25}$$

By (4.17) and (4.20) we have

$$\hat{\Gamma}^{\beta+n}_{\alpha;\gamma} - \hat{\Gamma}^{\beta+n}_{\alpha+n;\gamma+n} = 2\operatorname{Im}\big[\mathbb{G}^{\beta}_{\alpha\gamma} - \dot{\partial}_{\alpha}\dot{\partial}_{\gamma}\big(\overline{\mathbb{G}^{\beta}}\big)\big]. \tag{4.26}$$

It follows from (4.25) and (4.26) that

$$\operatorname{Im}\big\{\mathbb{G}^{\beta}_{\alpha\gamma}\big\} = \frac{1}{4}\big[\hat{\Gamma}^{\beta+n}_{\alpha;\gamma} - \hat{\Gamma}^{\beta+n}_{\alpha+n;\gamma+n} - \hat{\Gamma}^{\beta}_{\alpha+n;\gamma} - \hat{\Gamma}^{\beta}_{\alpha;\gamma+n}\big]. \tag{4.27}$$

By (4.24) and (4.27), we see that if F is a real Berwald metric on M, then both $\mathrm{Re}[\mathbb{G}^{\beta}_{\alpha\gamma}]$ and $\mathrm{Im}[\mathbb{G}^{\beta}_{\alpha\gamma}]$ are independent of the fiber coordinates u so that $\mathbb{G}^{\beta}_{\alpha\gamma}$ are independent of v. This implies that F is a weakly complex Berwald metric on M. □

Note that on a Kähler–Finsler manifold, the complex Berwald connection and the complex Rund connection coincide [15]. In this special case, our Definitions 4.5 and 4.3 coincide. Thus we have

Theorem 4.10. *Let F be a strongly convex Kähler–Finsler metric on a complex manifold M. Then F is a complex Berwald metric if and only if F is a real Berwald metric.*

Let $\Theta : \mathcal{V}^{1,0} \to \mathscr{H}^{1,0}$ denote the complex horizontal map associated to $\mathscr{H}^{1,0}$, and let $\hat{\Theta} : \mathcal{V}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}}$ denote the real horizontal map associated to $\mathcal{H}_{\mathbb{R}}$. Since ${}^{o} : \mathcal{V}^{1,0} \to \mathcal{V}_{\mathbb{R}}$ is a $\mathbb{R}$-isomorphism, we get a $\mathbb{R}$-isomorphism $\hat{}: \mathscr{H}^{1,0} \to \mathcal{H}_{\mathbb{R}}$ given by

$$\hat{H} = \hat{\Theta}\big(\big(\Theta^{-1}(H)\big)^{o}\big), \quad \forall H \in \mathscr{H}^{1,0}. \tag{4.28}$$

Note that $\widehat{\mathscr{X}_{\alpha}} = \hat{\delta}_{\alpha}$ and $\widehat{\sqrt{-1}\mathscr{X}_{\alpha}} = \hat{\delta}_{\alpha+n}$. As in Chapter 2 Section 2.6 in [1], we can establish similar theorems such as Theorems 2.6.6 and 2.6.8 in [1] with respect to the real and complex Berwald connections associated to a strongly convex weakly Kähler–Finsler metric F on a complex manifold M, just by replacing the Cartan and Chern–Finsler connections with the real and complex Berwald connections, respectively, and also $\mathcal{H}^{1,0}$ with $\mathscr{H}^{1,0}$. More precisely, we have

Proposition 4.11. *Let F be a strongly convex weakly Kähler–Finsler metric on a complex manifold M. If F is a weakly complex Berwald metric, then $\hat{\Theta}$ commutes with the complex structure J on $T^{1,0}M$ and*

$$\hat{H} = H^{o}, \quad \forall H \in \mathscr{H}^{1,0}. \tag{4.29}$$

Proof. As in the proof of Proposition 2.6.3 in [1], we can show that $\hat{\Theta}$ commutes with the complex structure J iff $\widehat{\sqrt{-1}H} = J\hat{H}$ for all $H \in \mathscr{H}^{1,0}$ iff $\mathcal{H}_{\mathbb{R}}$ is J-invariant iff

$$\hat{\Gamma}^{\beta}_{\gamma+n} = -\hat{\Gamma}^{\beta+n}_{\gamma} \quad \text{and} \quad \hat{\Gamma}^{\beta+n}_{\gamma+n} = \hat{\Gamma}^{\beta}_{\gamma} \tag{4.30}$$

for all $1 \leqslant \gamma, \beta \leqslant n$; $\hat{H} = H^{o}$ for all $H \in \mathscr{H}^{1,0}$ iff $\mathcal{H}_{\mathbb{R}}$ is J-invariant and

$$\mathbb{G}^{\beta}_{;\gamma} = \hat{\Gamma}^{\beta}_{\gamma} + \sqrt{-1}\hat{\Gamma}^{\beta+n}_{\gamma} \tag{4.31}$$

for all $1 \leqslant \gamma, \beta \leqslant n$.

If F is a strongly convex weakly Kähler–Finsler metric and also a weakly complex Berwald metric, then

$$\dot{\partial}_{\beta}\big(\overline{\mathbb{G}^{\beta}}\big) = 0. \tag{4.32}$$

Thus it follows from (4.9)–(4.12) that

$$\hat{\Gamma}^{\beta}_{\gamma+n} = -\,\mathrm{Im}\big[\mathbb{G}^{\beta}_{\gamma}\big] = -\hat{\Gamma}^{\beta+n}_{\gamma} \quad \text{and} \quad \hat{\Gamma}^{\beta+n}_{\gamma+n} = \mathrm{Re}\big[\mathbb{G}^{\beta}_{\gamma}\big] = \hat{\Gamma}^{\beta}_{\gamma}, \tag{4.33}$$

so that

$$\mathbb{G}^{\beta}_{;\gamma} = \mathrm{Re}\big[\mathbb{G}^{\beta}_{\gamma}\big] + \sqrt{-1}\,\mathrm{Im}\big[\mathbb{G}^{\beta}_{\gamma}\big] = \hat{\Gamma}^{\beta}_{\gamma} + \sqrt{-1}\hat{\Gamma}^{\beta+n}_{\gamma}. \tag{4.34}$$

This completes the proof. □

Let $\hat{\nabla}$ denote the covariant derivative associated to the real Berwald connection, and ∇ denote the covariant derivative associated to the complex Berwald connection. Since a complex Berwald connection is a good complex vertical connection, it follows by (2.2.7) in [1] that the $(2,0)$-torsion θ of the complex Berwald connection is

$$\theta = \frac{1}{2}\big[\mathscr{X}_{\mu}\big(\mathbb{G}^{\alpha}_{\nu}\big) - \mathscr{X}_{\nu}\big(\mathbb{G}^{\alpha}_{\mu}\big)\big]dz^{\mu} \wedge dz^{\nu} \otimes \dot{\partial}_{\alpha}, \tag{4.35}$$

and the $(1,1)$-torsion τ of the complex Berwald connection is given by

$$\tau = -\,\mathscr{X}_{\bar{\nu}}\big(\mathbb{G}^{\alpha}_{\mu}\big)\,dz^{\mu} \wedge d\overline{z^{\nu}} \otimes \dot{\partial}_{\alpha} - \dot{\partial}_{\bar{\beta}}\big(\mathbb{G}^{\alpha}_{\mu}\big)\,dz^{\mu} \wedge \overline{\phi^{\beta}} \otimes \dot{\partial}_{\alpha}, \tag{4.36}$$

where $\phi^{\beta} = dv^{\beta} + \mathbb{G}^{\beta}_{\alpha}\,dz^{\alpha}$.

Proposition 4.12. *Let F be a strongly convex weakly Kähler–Finsler metric on a complex manifold M. If F is a weakly complex Berwald metric, then*

$$\hat{\nabla}_{\widehat{\mathscr{X}}} V^o = (\nabla_{\mathscr{X}+\overline{\mathscr{X}}} V)^o - \frac{1}{2}\Theta^{-1}\big(\theta\big(\Theta(V), \chi\big)\big), \quad \forall V \in \mathcal{X}\big(\mathcal{V}^{1,0}\big). \tag{4.37}$$

Proof. Since for a weakly complex Berwald metric, we have $\dot{\partial}_{\bar{\beta}}(\mathbb{G}^{\alpha}_{\mu}) = 0$. Thus Proposition 4.12 follows immediately from Theorem 2.6.6 in [1]. □

Note that the non-vanishing components of the curvature operator Φ of the complex Berwald connection associated to (M, F) is given by

$$\Phi = \Phi^{\alpha}_{\beta} \otimes \big[dz^{\beta} \otimes \mathscr{X}_{\alpha} + \phi^{\beta} \otimes \dot{\partial}_{\alpha}\big], \tag{4.38}$$

where

$$\Phi^{\alpha}_{\beta} = \frac{1}{2}\Phi^{\alpha}_{\beta;\mu\nu}\, dz^{\mu} \wedge dz^{\nu} + \Phi^{\alpha}_{\beta;\mu\bar{\nu}}\, dz^{\mu} \wedge d\bar{z}^{\nu} + \Phi^{\alpha}_{\beta\nu;\mu}\, dz^{\mu} \wedge \phi^{\nu} + \Phi^{\alpha}_{\beta\bar{\nu};\mu}\, dz^{\mu} \wedge \overline{\phi^{\nu}},$$

and

$$\Phi^{\alpha}_{\beta;\mu\nu} = \mathscr{X}_{\mu}\big(\mathbb{G}^{\alpha}_{\beta\nu}\big) - \mathscr{X}_{\nu}\big(\mathbb{G}^{\alpha}_{\beta\mu}\big) + \mathbb{G}^{\gamma}_{\beta\nu}\mathbb{G}^{\alpha}_{\gamma\mu} - \mathbb{G}^{\gamma}_{\beta\mu}\mathbb{G}^{\alpha}_{\gamma\nu}, \tag{4.39}$$

$$\Phi^{\alpha}_{\beta;\mu\bar{\nu}} = -\mathscr{X}_{\bar{\nu}}\big(\mathbb{G}^{\alpha}_{\beta\mu}\big), \tag{4.40}$$

$$\Phi^{\alpha}_{\beta\bar{\nu};\mu} = -\dot{\partial}_{\bar{\nu}}\big(\mathbb{G}^{\alpha}_{\beta\mu}\big), \tag{4.41}$$

$$\Phi^{\alpha}_{\beta\nu;\mu} = -\dot{\partial}_{\nu}\big(\mathbb{G}^{\alpha}_{\beta\mu}\big). \tag{4.42}$$

It is clear that for a complex Berwald metric F, we have

$$\Phi^{\alpha}_{\beta\bar{\nu};\mu} = \Phi^{\alpha}_{\beta\nu;\mu} = 0.$$

Corollary 4.13. *Let F be a strongly convex weakly Kähler–Finsler metric on a complex manifold M. If F is a weakly complex Berwald metric with curvature components $\Phi^{\alpha}_{\beta;\mu\nu}$ satisfy*

$$\Phi^{\alpha}_{\beta;\mu\nu} v^{\beta} v^{\nu} = 0, \tag{4.43}$$

then

$$\hat{\nabla}_{\widehat{\mathscr{X}}} V^o = (\nabla_{\mathscr{X}+\overline{\mathscr{X}}} V)^o, \quad \forall V \in \mathcal{X}\big(\mathcal{V}^{1,0}\big). \tag{4.44}$$

Proof. It follows from (4.35) that

$$\theta\big(\Theta(V), \chi\big) = 0$$

if and only if

$$\big[\mathscr{X}_{\mu}\big(\mathbb{G}^{\alpha}_{\nu}\big) - \mathscr{X}_{\nu}\big(\mathbb{G}^{\alpha}_{\mu}\big)\big] v^{\nu} = 0. \tag{4.45}$$

Since

$$\begin{aligned}
\mathscr{X}_{\mu}\big(\mathbb{G}^{\alpha}_{\nu}\big) - \mathscr{X}_{\nu}\big(\mathbb{G}^{\alpha}_{\mu}\big) &= \mathscr{X}_{\mu}\big(\mathbb{G}^{\alpha}_{\beta\nu} v^{\beta}\big) - \mathscr{X}_{\nu}\big(\mathbb{G}^{\alpha}_{\beta\mu} v^{\beta}\big) \\
&= \mathscr{X}_{\mu}\big(\mathbb{G}^{\alpha}_{\beta\nu}\big) v^{\beta} - \mathbb{G}^{\gamma}_{\mu}\mathbb{G}^{\alpha}_{\gamma\nu} - \mathscr{X}_{\nu}\big(\mathbb{G}^{\alpha}_{\beta\mu}\big) v^{\beta} + \mathbb{G}^{\gamma}_{\nu}\mathbb{G}^{\alpha}_{\gamma\mu} \\
&= \big[\mathscr{X}_{\mu}\big(\mathbb{G}^{\alpha}_{\beta\nu}\big) - \mathbb{G}^{\gamma}_{\beta\mu}\mathbb{G}^{\alpha}_{\gamma\nu} - \mathscr{X}_{\nu}\big(\mathbb{G}^{\alpha}_{\beta\mu}\big) + \mathbb{G}^{\gamma}_{\beta\nu}\mathbb{G}^{\alpha}_{\gamma\mu}\big] v^{\beta} \\
&= \Phi^{\alpha}_{\beta;\mu\nu} v^{\beta},
\end{aligned}$$

Eq. (4.44) follows from Proposition 4.12 and (4.43). □

Corollary 4.14. *Let F be a strongly convex Kähler–Finsler metric on a complex manifold M. If F is a complex Berwald metric, then*

$$\hat{\nabla}_{\widehat{\mathscr{X}}} V^o = (\nabla_{\mathscr{X}+\overline{\mathscr{X}}} V)^o, \quad \forall V \in \mathcal{X}\big(\mathcal{V}^{1,0}\big). \tag{4.46}$$

Proof. Since on a Kähler–Finsler manifold, we have

$$\mathscr{X}_\alpha = \delta_\alpha, \qquad \mathbb{G}^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta;\mu} = \Gamma^\alpha_{\mu;\beta},$$

which implies that

$$\Phi^\alpha_{\beta;\mu\nu} = -K^\alpha_{\beta\nu\mu}.$$

Here $K^\alpha_{\beta\nu\mu}$ is called the second curvature tensor in [17] and it was proved in [17] that on a strongly pseudoconvex complex Finsler manifold $K^\alpha_{\beta\nu\mu} \equiv 0$. □

Proposition 4.15. *Let F be a strongly convex weakly Kähler–Finsler metric and also a weakly complex Berwald metric on a complex manifold M. Take $H \in \mathcal{H}_\mathbb{R}$ and $V \in \mathcal{X}(\mathcal{V}^{1,0})$.*

(i) *If V does not depend on the vector variables, i.e., V is the vertical lift of a vector field $\xi \in \mathcal{X}(T^{1,0}M)$, then*

$$\left\langle \hat{\nabla}_{\hat{H}} V^o - (\nabla_{H^o} V)^o \middle| \iota^o \right\rangle = 0; \tag{4.47}$$

(ii) *If $V \in \mathcal{X}(\mathcal{V}^{1,0})$ is any vertical vector field then*

$$\left\langle \hat{\nabla}_{\hat{H}} V^o - (\nabla_{H^o} V)^o \middle| \iota^o \right\rangle = 0. \tag{4.48}$$

Proof. This follows from (4.29) and Theorem 2.6.8 in [1]. □

By Propositions 4.11–4.12, Corollaries 4.13–4.14, Proposition 4.15, we immediately have the following theorem.

Theorem 4.16. *Let F be a strongly convex weakly Kähler–Finsler metric on a complex manifold M. If F is a weakly complex Berwald metric with the curvature components*

$$\Phi^\alpha_{\beta;\mu\nu} = \mathscr{X}_\mu\big(\mathbb{G}^\alpha_{\beta\nu}\big) - \mathscr{X}_\nu\big(\mathbb{G}^\alpha_{\beta\mu}\big) + \mathbb{G}^\gamma_{\beta\nu}\mathbb{G}^\alpha_{\gamma\mu} - \mathbb{G}^\gamma_{\beta\mu}\mathbb{G}^\alpha_{\gamma\nu} \tag{4.49}$$

satisfying

$$\Phi^\alpha_{\beta;\mu\nu} v^\beta v^\nu = 0, \tag{4.50}$$

then the real and complex Berwald connections associated to F coincide.

Theorem 4.17. *Let F be a strongly convex Kähler–Finsler metric on a complex manifold M. If F is a complex Berwald metric, then the real and complex Berwald connections associated to F coincide.*

Proof. This follows from Theorem 4.16, since on a Kähler–Finsler manifold the curvature components

$$\Phi^\alpha_{\beta;\mu\nu} = -K_\beta{}^\alpha{}_{\mu\nu},$$

where $K_\beta{}^\alpha{}_{\mu\nu}$ is the second curvature tensor and it was proved in [17] that

$$K_\beta{}^\alpha{}_{\mu\nu} \equiv 0. \qquad \square$$

5. Complex Wrona metric

In this section, we shall introduce the complex Wrona metric [10] in $\mathbb{C}^n$ $(n \geqslant 2)$, which severs as a good example of weakly complex Berwald metric but not a complex Berwald metric.

Let $M = \mathbb{C}^n$, $n \geqslant 2$, be the complex Euclidean space with the canonical Hermitian inner product $\langle z, v\rangle = \sum_{\alpha=1}^n z^\alpha \overline{v^\alpha}$ and the norm $\|z\| = \sqrt{\langle z, z\rangle}$. We denote by O the origin of $\mathbb{C}^n$, $T^{1,0}M = \mathbb{C}^n \times \mathbb{C}^n$ the holomorphic tangent bundle of M, and $\tilde{M}$ be complement of the zero section in $T^{1,0}M$. Let P and Q be two points in $\mathbb{C}^n$ and l be the line which passes through the points P and Q. Assume that l does not pass through the origin O of $\mathbb{C}^n$. Denote H the projection point of the origin O onto l and define

$$F(z, v) = \frac{|PQ|}{|OH|}, \tag{5.1}$$

where $|PQ|$ and $|OH|$ denote the Euclidean lengths of the segments PQ and OH, respectively. It is easy to check that $H = z - \langle z, v\rangle \|v\|^{-2} v$ so that

$$F(z,v)=\frac{\|v\|^2}{\sqrt{\|z\|^2\|v\|^2-|\langle z,v\rangle|^2}},\quad (z,v)\in\Omega, \tag{5.2}$$

where

$$\Omega=\bigl\{(z,v)\in\mathbb{C}^n\times\mathbb{C}^n:\ z\neq\lambda v,\ \lambda\in\mathbb{C}\bigr\}.$$

The metric F defined by (5.2) is called the complex Wrona metric in [10].

6. The holomorphic curvature of the complex Wrona metric

In this section we shall use the techniques developed in [1] to investigate the properties of F since it is smooth over the open subset $\Omega\subset\tilde{M}$.

Now we denote by $G=F^2$ and $A(z,v)=\|z\|^2\|v\|^2-|\langle z,v\rangle|^2$. It is clear that the function A is symmetric with respect to z,v, and it enjoys the $(1,1)$-homogeneity property, i.e.,

$$A(\lambda z,v)=\lambda\bar{\lambda}A(z,v),\qquad A(z,\lambda v)=\lambda\bar{\lambda}A(z,v) \tag{6.1}$$

for every $\lambda\in\mathbb{C}$ with $\lambda\neq 0$. In the following we denote the derivatives of A with respect to the v-coordinates by

$$A_\alpha=\frac{\partial A}{\partial v^\alpha},\qquad A_{\bar\alpha}=\frac{\partial A}{\partial\overline{v^\alpha}};$$

and the derivatives of A with respect to z-coordinates by indexes after a semicolon, i.e.,

$$A_{;\beta}=\frac{\partial A}{\partial z^\beta},\qquad A_{;\bar\beta}=\frac{\partial A}{\partial\overline{z^\beta}},\qquad A_{\bar\beta;\gamma}=\frac{\partial^2 A}{\partial z^\gamma\partial\overline{v^\beta}}.$$

Using the $(1,1)$-homogeneity property (6.1) of A, we immediately have

$$A_\beta v^\beta=A_{\bar\beta}\overline{v^\beta}=A_{;\beta}z^\beta=A_{;\bar\beta}\overline{z^\beta}=A,\qquad A_\beta z^\beta=A_{\bar\beta}\overline{z^\beta}=A_{;\beta}v^\beta=A_{;\bar\beta}\overline{v^\beta}=0. \tag{6.2}$$

Similarly, it is easy to check that

$$\sum_{\beta=1}^n A_{;\beta}A_{;\bar\beta}=\|v\|^2A,\qquad \sum_{\beta=1}^n A_\beta A_{\bar\beta}=\|z\|^2A,\qquad \sum_{\beta=1}^n A_{;\beta}A_{\bar\beta}=-\overline{\langle z,v\rangle}A, \tag{6.3}$$

$$\sum_{\beta=1}^n A_{\bar\beta;\gamma}\overline{z^\beta}=0,\qquad \sum_{\beta=1}^n A_{\bar\beta;\gamma}v^\gamma=0,\qquad \sum_{\beta=1}^n A_{\bar\beta;\gamma}A_{;\beta}=-\overline{\langle z,v\rangle}A_{;\gamma}, \tag{6.4}$$

$$\sum_{\alpha=1}^n A_{\bar\alpha;\beta}A_\alpha=\|z\|^2A_{;\beta},\qquad \sum_{\mu=1}^n A_{\bar\mu;\mu}=(1-n)\overline{\langle z,v\rangle}. \tag{6.5}$$

Lemma 6.1. *(See [11].) Suppose a nonsingular $n\times n$ matrix C has a known inverse C^{-1}, $Z,Y\in\mathbb{C}^n$ are two column vectors and Y^* is the Hermitian transpose of Y. If*

$$B=C+cZY^*$$

is nonsingular for a constant c, then

$$|B|==|C|\bigl|I+cC^{-1}ZY^*\bigr|=|C|\bigl(1+cY^*C^{-1}Z\bigr), \tag{6.6}$$

$$B^{-1}=C^{-1}-\frac{c}{1+cY^*C^{-1}Z}C^{-1}ZY^*C^{-1}, \tag{6.7}$$

where $|B|$ denote the determinant of the matrix B.

The following proposition shows that under suitable assumption, F is strongly pseudoconvex with respect to v in a subset $\mathcal{D}\subset\Omega$.

Proposition 6.2. *Let F be the complex Wrona metric which is defined by (5.2). Then*

$$G_{\alpha\overline{\beta}}=\frac{\|v\|^2(2A-\|z\|^2\|v\|^2)}{A^2}\delta_{\alpha\overline{\beta}}+\frac{2|\langle z,v\rangle|^2}{A^3}A_{;\alpha}A_{;\overline{\beta}}+\frac{\|v\|^4}{A^2}\overline{z^\alpha}z^\beta, \tag{6.8}$$

$$G^{\overline{\beta}\alpha}=\frac{A^2}{\|v\|^2(2A-\|z\|^2\|v\|^2)}\left[\delta^{\overline{\beta}\alpha}-\frac{2|\langle z,v\rangle|^2}{\|v\|^2A^2}A_{;\beta}A_{;\overline{\alpha}}-\frac{\|z\|^2\|v\|^4}{2A^2}\overline{z^\beta}z^\alpha+\frac{|\langle z,v\rangle|^2}{A^2}\overline{z^\beta}A_{;\overline{\alpha}}+\frac{|\langle z,v\rangle|^2}{A^2}A_{;\beta}z^\alpha\right]. \tag{6.9}$$

Moreover, F is strongly pseudoconvex with respect to v in the subset $\mathcal{D}$, where

$$\mathcal{D}=\left\{(z,v)\in\Omega\;\middle|\;\cos\theta:=\frac{|\langle z,v\rangle|}{\|z\|\cdot\|v\|}<\frac{\sqrt{2}}{2}\right\}. \tag{6.10}$$

Proof. Differential (5.2) with respect to v^α, we get

$$G_\alpha=\frac{\|v\|^2}{A}\left(2\overline{v^\alpha}-\frac{\|v\|^2}{A}A_\alpha\right). \tag{6.11}$$

Differential (6.11) with respect to $\overline{v^\beta}$, we get (6.8). In order to obtain (6.9), we denote $z:=(z^1,\dots,z^n)$, $X:=(A_{;\overline{1}},\dots,A_{;\overline{n}})$. Let z^* and X^* be the Hermitian transposes of z and X, respectively. Then it is easy to check that

$$XX^*=\|v\|^2A,\qquad Xz^*=zX^*=A,\qquad zz^*=\|z\|^2,\qquad zX^*Xz^*=A^2. \tag{6.12}$$

With this notions, the Hermitian matrix $(G_{\alpha\overline{\beta}})$ can be rewritten as

$$(G_{\alpha\overline{\beta}})=\frac{\|v\|^2(2A-\|z\|^2\|v\|^2)}{A^2}I+\frac{2|\langle z,v\rangle|^2}{A^3}X^*X+\frac{\|v\|^4}{A^2}z^*z, \tag{6.13}$$

where I is the n by n identity matrix. Using Eq. (6.7) in Lemma 6.1, together with Eqs. (6.2)–(6.4) and (6.12), we obtain

$$\left(G^{\overline{\beta}\alpha}\right)=\frac{A^2}{\|v\|^2(2A-\|z\|^2\|v\|^2)}\left[I-\frac{2|\langle z,v\rangle|^2}{\|v\|^2A^2}X^*X-\frac{\|z\|^2\|v\|^4}{2A^2}z^*z+\frac{|\langle z,v\rangle|^2}{A^2}\left(z^*X+X^*z\right)\right], \tag{6.14}$$

or equivalently, we get (6.9). It follows immediately from (6.8) that $(G_{\alpha\overline{\beta}})$ is positive definite over the subset $\mathcal{D}\subset T^{1,0}M$. □

Proposition 6.3. *Let $\varGamma^\alpha_{;\mu}$ be the Chern–Finsler nonlinear connection coefficients associated to F. Then*

$$\varGamma^\alpha_{;\mu}v^\mu=0,\quad\forall(z,v)\in\mathcal{D}. \tag{6.15}$$

Proof. It is known that the Chern–Finsler nonlinear connection coefficients $\varGamma^\alpha_{;\mu}$ associated to a strongly pseudoconvex complex Finsler metric F is given by (3.2), i.e.,

$$\varGamma^\alpha_{;\mu}=G^{\overline{\beta}\alpha}G_{\overline{\beta};\mu}. \tag{6.16}$$

Differential G with respect to $\overline{v^\beta}$ and z^μ, we get

$$G_{\overline{\beta};\mu}=-\frac{2\|v\|^2}{A^2}v^\beta A_{;\mu}-\frac{\|v\|^4}{A^2}A_{\overline{\beta};\mu}+\frac{2\|v\|^4}{A^3}A_{\overline{\beta}}A_{;\mu}. \tag{6.17}$$

Thus by (6.2) and (6.4), we have

$$G_{\overline{\beta};\gamma}v^\gamma=0.$$

Consequently,

$$\varGamma^\alpha_{;\mu}v^\mu=G^{\overline{\beta}\alpha}G_{\overline{\beta};\mu}v^\mu=0,$$

which completes the proof. □

Now we are able to obtain the explicit expression of $\varGamma^\alpha_{;\mu}$.

Proposition 6.4. *The Chern–Finsler nonlinear connection coefficients $\varGamma^\alpha_{;\mu}$ associated to the complex Wrona metric F is given by*

$$\varGamma^\alpha_{;\mu}=\frac{A^2}{2A-\|z\|^2\|v\|^2}\left\{-\frac{2}{A^2}v^\alpha A_{;\mu}-\frac{\|v\|^2}{A^2}A_{\overline{\alpha};\mu}+\frac{2\|v\|^2}{A^3}A_{\overline{\alpha}}A_{;\mu}+\frac{\|v\|^2}{A^3}\overline{\langle z,v\rangle}z^\alpha A_{;\mu}\right\}. \tag{6.18}$$

Proof. This is a direct calculation. In fact, substituting (6.9) and (6.17) into (6.16), we get

$$\begin{aligned}\Gamma^{\alpha}_{;\mu} = \frac{A^2}{\|v\|^2[2A-\|z\|^2\|v\|^2]}\sum_{\beta=1}^{n}\bigg\{&-\frac{2\|v\|^2}{A^2}v^\alpha A_{;\mu}-\frac{\|v\|^4}{A^2}A_{\overline{\alpha};\mu}+\frac{2\|v\|^4}{A^3}A_{\overline{\alpha}}A_{;\mu}+\frac{4|\langle z,v\rangle|^2}{A^4}A_{;\beta}v^\beta A_{;\overline{\alpha}}A_{;\mu}\\ &+\frac{2\|v\|^2|\langle z,v\rangle|^2}{A^4}A_{;\beta}A_{\overline{\beta};\mu}A_{;\overline{\alpha}}-\frac{4\|v\|^2|\langle z,v\rangle|^2}{A^5}A_{;\beta}A_{\overline{\beta}}A_{;\overline{\alpha}}A_{;\mu}+\frac{\|z\|^2\|v\|^6}{A^4}\overline{z^\beta}v^\beta z^\alpha A_{;\mu}\\ &+\frac{\|z\|^2\|v\|^8}{2A^4}\overline{z^\beta}A_{\overline{\beta};\mu}z^\alpha-\frac{\|z\|^2\|v\|^8}{A^5}\overline{z^\beta}A_{\overline{\beta}}A_{;\mu}z^\alpha-\frac{2\|v\|^2|\langle z,v\rangle|^2}{A^4}v^\beta A_{;\mu}\big(\overline{z^\beta}A_{;\overline{\alpha}}+A_{;\beta}z^\alpha\big)\\ &-\frac{\|v\|^4|\langle z,v\rangle|^2}{A^4}A_{\overline{\beta};\mu}\big(\overline{z^\beta}A_{;\overline{\alpha}}+A_{;\beta}z^\alpha\big)+\frac{2\|v\|^4|\langle z,v\rangle|^2}{A^5}A_{\overline{\beta}}A_{;\mu}\big(\overline{z^\beta}A_{;\overline{\alpha}}+A_{;\beta}z^\alpha\big)\bigg\}.\end{aligned}$$

Using Eqs. (6.2)–(6.4), the above equation reduces to

$$\begin{aligned}\Gamma^{\alpha}_{;\mu} = \frac{A^2}{2A-\|z\|^2\|v\|^2}\bigg\{&-\frac{2}{A^2}v^\alpha A_{;\mu}-\frac{\|v\|^2}{A^2}A_{\overline{\alpha};\mu}+\frac{2\|v\|^2}{A^3}A_{\overline{\alpha}}A_{;\mu}-\frac{2|\langle z,v\rangle|^2}{A^4}\overline{\langle z,v\rangle}A_{;\overline{\alpha}}A_{;\mu}\\ &+\frac{4|\langle z,v\rangle|^2}{A^4}\overline{\langle z,v\rangle}A_{;\overline{\alpha}}A_{;\mu}+\frac{\|z\|^2\|v\|^4}{A^4}\overline{\langle z,v\rangle}z^\alpha A_{;\mu}-\frac{2|\langle z,v\rangle|^2}{A^4}\overline{\langle z,v\rangle}A_{;\overline{\alpha}}A_{;\mu}\\ &+\frac{\|v\|^2|\langle z,v\rangle|^2}{A^4}\overline{\langle z,v\rangle}z^\alpha A_{;\mu}-\frac{2\|v\|^2|\langle z,v\rangle|^2}{A^4}\overline{\langle z,v\rangle}z^\alpha A_{;\mu}\bigg\}.\end{aligned}$$

Rearrange the terms in the above equation we obtain (6.18), this completes the proof. □

Proposition 6.5. *Let $\Gamma^{\alpha}_{\beta;\mu}$ be the horizontal connection coefficients of the Chern–Finsler connection that associated to the complex Wrona metric F. Then*

$$G_\alpha\big(\Gamma^{\alpha}_{\mu;\beta}-\Gamma^{\alpha}_{\beta;\mu}\big)v^\mu = 2G_{;\beta}.$$

Proof. Note that the Chern–Finsler nonlinear connection coefficients $\Gamma^{\alpha}_{;\mu}$ and the horizontal Chern–Finsler connection coefficients $\Gamma^{\alpha}_{\beta;\mu}$ that associated to F satisfy (3.6). Thus differential (6.15) with respect to v^β and using (3.6), we obtain

$$\Gamma^{\alpha}_{\beta;\mu}v^\mu = -\Gamma^{\alpha}_{;\beta} = -\Gamma^{\alpha}_{\mu;\beta}v^\mu,$$

from which we have

$$\big(\Gamma^{\alpha}_{\mu;\beta}-\Gamma^{\alpha}_{\beta;\mu}\big)v^\mu = 2\Gamma^{\alpha}_{;\beta}.$$

Thus it follows from (6.11) and (6.18) that

$$\begin{aligned}G_\alpha\big(\Gamma^{\alpha}_{\mu;\beta}-\Gamma^{\alpha}_{\beta;\mu}\big)v^\mu = -\frac{2\|v\|^2A}{2A-\|z\|^2\|v\|^2}\sum_{\alpha=1}^{n}\bigg(2\overline{v^\alpha}-\frac{\|v\|^2}{A}A_\alpha\bigg)\bigg\{&\frac{2}{A^2}v^\alpha A_{;\beta}+\frac{\|v\|^2}{A^2}A_{\overline{\alpha};\beta}\\ &-\frac{2\|v\|^2}{A^3}A_{\overline{\alpha}}A_{;\beta}-\frac{\|v\|^2}{A^3}\overline{\langle z,v\rangle}z^\alpha A_{;\beta}\bigg\}.\end{aligned}$$

Substituting (6.2)–(6.5) into the above equation, we obtain

$$G_\alpha\big(\Gamma^{\alpha}_{\mu;\beta}-\Gamma^{\alpha}_{\beta;\mu}\big)v^\mu = -\frac{2\|v\|^2A}{2A-\|z\|^2\|v\|^2}\cdot\frac{\|v\|^2(2A-\|z\|^2\|v\|^2)}{A^3}A_{;\beta} = -\frac{2\|v\|^4}{A^2}A_{;\beta} = 2G_{;\beta},$$

which completes the proof. □

Proposition 6.5 shows that the complex Wrona metric F is not a weakly Kähler–Finsler metric in $\mathbb{C}^n$ in the sense of [1].

In [1, cf. p. 108], the holomorphic curvature K_F of the Chern–Finsler connection associated to a strongly pseudoconvex complex Finsler metric F is defined by

$$K_F(z,v) = -\frac{2}{G^2}G_\alpha\delta_{\overline{\nu}}\big(\Gamma^{\alpha}_{\mu}\big)v^\mu\overline{v^\nu}.$$

In [15, cf. p. 78], the Ricci scalar curvature Ric_F of the Chern–Finsler connection associated to a strongly pseudoconvex complex Finsler metric F is defined by

$$Ric_F(z,v) = -\delta_{\overline{\alpha}}\big(\Gamma^{\mu}_{;\mu}\big)\overline{v^{\alpha}} = -\overline{\chi}\big(\Gamma^{\mu}_{;\mu}\big),$$

where the index μ is summed from 1 to n.

Replacing the Hermitian inner product $\langle\cdot,\cdot\rangle$ in [1] with $\langle\cdot,\cdot\rangle_{\mathbf{b}}$, $\delta_{\overline{\alpha}}$ with $\mathscr{X}_{\overline{\alpha}}$ and $\Gamma^{\alpha}_{;\mu}$ with $\mathbb{G}^{\alpha}_{\mu}$, we immediately obtain the following definition of holomorphic curvature and Ricci scalar curvature with respect to the complex Berwald connection.

Definition 6.6. The holomorphic curvature K and the Ricci scalar curvature Ric of the complex Berwald connection that associated to a strongly pseudoconvex complex Finsler metric F are defined respectively by

$$K(z,v) = -\frac{2}{G^2} G_{\alpha}\mathscr{X}_{\overline{\nu}}\big(\mathbb{G}^{\alpha}_{\mu}\big)v^{\mu}\overline{v^{\nu}}, \tag{6.19}$$

$$Ric(z,v) = -\mathscr{X}_{\overline{\alpha}}\big(\mathbb{G}^{\mu}_{\mu}\big)\overline{v^{\alpha}}, \tag{6.20}$$

where the index μ is summed from 1 to n.

Theorem 6.7. *The complex Wrona metric*

$$F(z,v) = \frac{\|v\|^2}{\sqrt{\|z\|^2\|v\|^2 - |\langle z,v\rangle|^2}}, \quad \forall (z,v)\in\mathcal{D} \tag{6.21}$$

in $\mathbb{C}^n$ is a weakly complex Berwald metric whose holomorphic curvature and Ricci scalar curvature vanish identically, i.e.,

$$K(z,v)\equiv 0, \qquad Ric(z,v)\equiv 0. \tag{6.22}$$

Proof. By Proposition 6.3,

$$\mathbb{G}^{\alpha} = \frac{1}{2}\Gamma^{\alpha}_{;\mu}v^{\mu} = 0, \quad \forall (z,v)\in\mathcal{D}. \tag{6.23}$$

Thus $\mathbb{G}^{\alpha}_{\beta\mu} = \dot{\partial}_{\beta}\dot{\partial}_{\mu}(\mathbb{G}^{\alpha}) \equiv 0$ which clearly implies that $\mathbb{G}^{\alpha}_{\beta\mu}$ are independent of fiber coordinates v. So that the complex Wrona metric (6.21) is a weakly complex Berwald metric. Next we are in a position to show that the complex Wrona metric is not a complex Berwald metric. We prove this by contradiction. If, on the contrary, the complex Wrona metric is a complex Berwald metric, i.e., the horizontal Chern–Finsler connection coefficients $\Gamma^{\alpha}_{\beta;\mu}$ are independent of fiber coordinate v, then it follows that $\Gamma^{\alpha}_{;\beta} = \Gamma^{\alpha}_{\beta;\mu}(z)v^{\beta}$ is linear with respect to the fiber coordinates v. This contradicts with the explicit expression of $\Gamma^{\alpha}_{;\beta}$ in Proposition 6.4.

For the holomorphic curvature and Ricci scalar curvature of the complex Wrona metric. By Definition 6.6,

$$K(z,v) = -\frac{2}{G^2}G_{\alpha}\mathscr{X}_{\overline{\nu}}\big(\mathbb{G}^{\alpha}_{\mu}\big)v^{\mu}\overline{v^{\nu}} = -\frac{2}{G^2}G_{\alpha}\mathscr{X}_{\overline{\nu}}\big(\mathbb{G}^{\alpha}\big)\overline{v^{\nu}} = 0,$$

where in last equality of the above equation we used (6.23). Next, it is clear that

$$Ric = -\mathscr{X}_{\overline{\alpha}}\big(\mathbb{G}^{\mu}_{\mu}\big)\overline{v^{\alpha}} = 0, \tag{6.24}$$

since by (6.23) we have

$$\mathbb{G}^{\alpha}_{\mu} = \dot{\partial}_{\mu}\big(\mathbb{G}^{\alpha}\big) = 0. \qquad \square$$

Remark 6.8. Note that the holomorphic curvature K_F of the Chern–Finsler connection associated to (6.21) is also vanishes, i.e.,

$$K_F(z,v)\equiv 0. \tag{6.25}$$

The Ricci scalar curvature Ric_F of the Chern–Finsler connection associated to (6.21) is

$$Ric_F(z,v)\equiv 0 \tag{6.26}$$

for $n=2$, and

$$Ric_F(z,v) = \frac{(2-n)(\|z\|^2\|v\|^2 - |\langle z,v\rangle|^2)\|v\|^4}{(\|z\|^2\|v\|^2 - 2|\langle z,v\rangle|^2)^2} < 0 \tag{6.27}$$

for $n>2$.

In fact, by (6.18) and (6.2)–(6.5) we get

$$\Gamma^{\mu}_{;\mu} = \frac{A^2}{2A - \|z\|^2\|v\|^2} \sum_{\mu=1}^{n} \left\{ -\frac{2}{A^2} v^{\mu} A_{;\mu} - \frac{\|v\|^2}{A^2} A_{\overline{\mu};\mu} + \frac{2\|v\|^2}{A^3} A_{\overline{\mu}} A_{;\mu} + \frac{\|v\|^2}{A^3} \overline{\langle z, v\rangle} z^{\mu} A_{;\mu} \right\}$$
$$= \frac{(n-2)\|v\|^2 \overline{\langle z, v\rangle}}{2A - \|z\|^2\|v\|^2}. \tag{6.28}$$

On the other hand, using (6.15) we have

$$\overline{v^{\alpha}} \delta_{\overline{\alpha}} = \overline{v^{\alpha}} \left(\partial_{\overline{\alpha}} - \overline{\Gamma^{\gamma}_{;\alpha}} \dot{\partial}_{\overline{\gamma}} \right) = \overline{v^{\alpha}} \partial_{\overline{\alpha}}. \tag{6.29}$$

Therefore it follows from (6.28) and (6.29) that

$$\mathrm{Ric}_F = -\delta_{\overline{\alpha}} \left(\Gamma^{\mu}_{;\mu} \right) \overline{v^{\alpha}} = -\overline{v^{\alpha}} \partial_{\overline{\alpha}} \left(\Gamma^{\mu}_{;\mu} \right) = \frac{(2-n)A\|v\|^4}{(2A - \|z\|^2\|v\|^2)^2}.$$

7. The real geodesic of the complex Wrona metric

In this section, we shall investigate the real geodesic of the complex Wrona metric (6.21), which was posed as an open problem in [10]. We derive the geodesic equation for F and obtain explicitly the real geodesic of F, which lies on the Euclidean sphere

$$\mathbf{S}^{2n-1} = \left\{ z \in \mathbb{C}^n \;\middle|\; \|z\| = 1 \right\} \subset \mathbb{C}^n.$$

In [10], the authors proved that if any geodesic of the complex Wrona metric F on $\mathbf{S}^{2n-1}$ ($n \geqslant 2$) is parameterized by arc length t with $0 < t < \frac{\pi}{2}$, then the length $L(\sigma) = t$. In this paper, we obtain the following theorem.

Theorem 7.1. *Let F be the complex Wrona metric in $\mathbb{C}^n$ and $\sigma(t) = (\sigma^1(t), \ldots, \sigma^n(t))$ be a geodesic of F. Then σ satisfies the following equation*:

$$\left(\|\sigma\|^2\|\dot{\sigma}\|^2 - \left| \langle \sigma, \dot{\sigma} \rangle \right|^2 \right) \ddot{\sigma}^{\alpha} = 2\|\dot{\sigma}\|^2 \langle \sigma, \dot{\sigma} \rangle \dot{\sigma}^{\alpha} - \|\dot{\sigma}\|^4 \sigma^{\alpha}, \quad \alpha = 1, \ldots, n. \tag{7.1}$$

For any given points $p, q \in \mathbf{S}^{2n-1}$ with $\langle p, q \rangle = 0$, there exists a unique closed geodesic

$$\sigma(t) = \frac{1}{2}\left[\left(p - \sqrt{-1}q \right) e^{\sqrt{-1}t} + \left(p + \sqrt{-1}q \right) e^{-\sqrt{-1}t} \right], \quad t \in \mathbb{R} \tag{7.2}$$

on $\mathbf{S}^{2n-1}$ such that $\sigma(0) = p$, $\dot{\sigma}(0) = q$ and $\sigma, \dot{\sigma} \in \mathbf{S}^{2n-1}$ with $\langle \sigma, \dot{\sigma} \rangle = 0$. Moreover, the arc length $L(\sigma)$ of σ satisfies

$$L(\sigma) = 2\pi. \tag{7.3}$$

Proof. In local coordinates, σ satisfies [1, p. 101]:

$$\ddot{\sigma}^{\alpha} + \Gamma^{\alpha}_{;\mu} \dot{\sigma}^{\mu} = G^{\overline{\nu}\alpha} G_{\beta\overline{\gamma}} \left(\overline{\Gamma^{\gamma}_{\mu;\nu}} - \overline{\Gamma^{\gamma}_{\nu;\mu}} \right) \dot{\sigma}^{\beta} \overline{\dot{\sigma}^{\mu}}.$$

Note that since $\Gamma^{\alpha}_{;\mu} \dot{\sigma}^{\mu} = 0$ and

$$G_{\beta\overline{\gamma}} \left(\overline{\Gamma^{\gamma}_{\mu;\nu}} - \overline{\Gamma^{\gamma}_{\nu;\mu}} \right) \dot{\sigma}^{\beta} \overline{\dot{\sigma}^{\mu}} = G_{\overline{\gamma}} \left(\overline{\Gamma^{\gamma}_{\mu;\nu}} - \overline{\Gamma^{\gamma}_{\nu;\mu}} \right) \overline{\dot{\sigma}^{\mu}} = 2G_{;\overline{\nu}},$$

we have

$$\ddot{\sigma}^{\alpha} = 2G^{\overline{\nu}\alpha} G_{;\overline{\nu}}.$$

Thus by (6.9), we have

$$\ddot{\sigma}^{\alpha} = \frac{2A^2}{\|\dot{\sigma}\|^2 [2A - \|\sigma\|^2\|\dot{\sigma}\|^2]} \sum_{\nu=1}^{n} \left(\delta^{\overline{\nu}\alpha} - \frac{2|\langle \sigma, \dot{\sigma} \rangle|^2}{A^2\|\dot{\sigma}\|^2} A_{;\nu} A_{;\overline{\alpha}} - \frac{\|\sigma\|^2\|\dot{\sigma}\|^4}{2A^2} \overline{\sigma^{\nu}} \sigma^{\alpha} \right.$$
$$\left. + \frac{|\langle \sigma, \dot{\sigma} \rangle|^2}{A^2} \overline{\sigma^{\nu}} A_{;\overline{\alpha}} + \frac{|\langle \sigma, \dot{\sigma} \rangle|^2}{A^2} A_{;\nu} \sigma^{\alpha} \right) \cdot \left(-\frac{\|\dot{\sigma}\|^4}{A^2} A_{;\overline{\nu}} \right).$$

That is,

$$\ddot{\sigma}^\alpha = -\frac{2\|\dot{\sigma}\|^2}{2A - \|\sigma\|^2\|\dot{\sigma}\|^2}\sum_{\nu=1}^n\bigg(A_{;\bar{\alpha}} - \frac{2|\langle\sigma,\dot{\sigma}\rangle|^2}{A^2\|\dot{\sigma}\|^2}A_{;\nu}A_{;\bar{\nu}}A_{;\bar{\alpha}} - \frac{\|\sigma\|^2\|\dot{\sigma}\|^4}{2A^2}\overline{\sigma^\nu}A_{;\bar{\nu}}\sigma^\alpha$$
$$+ \frac{|\langle\sigma,\dot{\sigma}\rangle|^2}{A^2}\overline{\sigma^\nu}A_{;\bar{\nu}}A_{;\bar{\alpha}} + \frac{|\langle\sigma,\dot{\sigma}\rangle|^2}{A^2}A_{;\nu}A_{;\bar{\nu}}\sigma^\alpha\bigg).$$

Using (6.2)–(6.5) and arranging the resulted terms, we obtain

$$\ddot{\sigma}^\alpha = -\frac{2\|\dot{\sigma}\|^2}{2A - \|\sigma\|^2\|\dot{\sigma}\|^2}\bigg(\frac{2A - \|\sigma\|^2\|\dot{\sigma}\|^2}{A}A_{;\bar{\alpha}} - \frac{(2A - \|\sigma\|^2\|\dot{\sigma}\|^2)\|\dot{\sigma}\|^2}{2A}\sigma^\alpha\bigg).$$

Substituting $A_{;\bar{\alpha}} = \|\dot{\sigma}\|^2\sigma^\alpha - \langle\sigma,\dot{\sigma}\rangle\dot{\sigma}^\alpha$ into the above equation, we get the following geodesic equation of the complex Wrona metric F:

$$\ddot{\sigma}^\alpha = -\frac{\|\dot{\sigma}\|^4}{A}\sigma^\alpha + \frac{2\|\dot{\sigma}\|^2\langle\sigma,\dot{\sigma}\rangle}{A}\dot{\sigma}^\alpha, \quad \alpha = 1,\dots,n. \tag{7.4}$$

Substituting $A(\sigma,\dot{\sigma}) = \|\sigma\|^2\|\dot{\sigma}\|^2 - |\langle\sigma,\dot{\sigma}\rangle|^2$ into the above equation we get (7.1).

Let $p, q \in \mathbf{S}^{2n-1}$ with $\langle p,q\rangle = 0$. Suppose that σ is a geodesic of F such that σ, $\dot{\sigma} \in \mathbf{S}^{2n-1}$ and $\langle\sigma,\dot{\sigma}\rangle = 0$. We need to recover the explicit formula for σ. By assumption, the geodesic σ to be determined satisfies

$$\|\sigma\|^2 = \|\dot{\sigma}\|^2 = 1, \tag{7.5}$$

$$A(\sigma,\dot{\sigma}) = \|\sigma\|^2\|\dot{\sigma}\|^2 - \big|\langle\sigma,\dot{\sigma}\rangle\big|^2 = 1. \tag{7.6}$$

Substituting (7.5) and (7.6) into the geodesic equation (7.4), we get

$$\ddot{\sigma}^\alpha + \sigma^\alpha = 0, \quad \alpha = 1,\dots,n. \tag{7.7}$$

It is clear that the general solution of (7.7) is given by

$$\sigma^\alpha = c_1e^{\sqrt{-1}t} + c_2e^{-\sqrt{-1}t}, \tag{7.8}$$

where c_1, c_2 are constants to be determined. Substituting the initial conditions $\sigma(0) = p, \dot{\sigma}(0) = q$ into (7.8), we get

$$c_1 = \frac{1}{2}\big(p - \sqrt{-1}q\big), \qquad c_2 = \frac{1}{2}\big(p + \sqrt{-1}q\big).$$

Thus we get (7.2).

Next we shall show that the geodesic σ given by (7.2) actually satisfies $\|\sigma\|^2 = \|\dot{\sigma}\|^2 = 1$ and $\langle\sigma,\dot{\sigma}\rangle = 0$. Differentiating (7.2) with respect to t, we get

$$\dot{\sigma}(t) = \frac{\sqrt{-1}}{2}\big[\big(p - \sqrt{-1}q\big)e^{\sqrt{-1}t} - \big(p + \sqrt{-1}q\big)e^{-\sqrt{-1}t}\big]. \tag{7.9}$$

By assumption,

$$\|p\|^2 = \|q\|^2 = 1, \qquad \langle p,q\rangle = \overline{\langle q,p\rangle} = 0. \tag{7.10}$$

A direction calculation by using (7.2), (7.9) and (7.10) gives $\|\sigma\|^2 = \|\dot{\sigma}\|^2 = 1$. Similarly, by (7.10) we obtain

$$\begin{aligned}\langle\sigma,\dot{\sigma}\rangle &= -\frac{\sqrt{-1}}{4}\big[\big\langle\big(p - \sqrt{-1}q\big),\big(p - \sqrt{-1}q\big)\big\rangle - e^{2\sqrt{-1}t}\big\langle\big(p - \sqrt{-1}q\big),\big(p + \sqrt{-1}q\big)\big\rangle \\ &\quad + e^{-2\sqrt{-1}t}\big\langle\big(p + \sqrt{-1}q\big),\big(p - \sqrt{-1}q\big)\big\rangle - \big\langle\big(p + \sqrt{-1}q\big),\big(p + \sqrt{-1}q\big)\big\rangle\big] \\ &= -\frac{\sqrt{-1}}{4}\big\{2\sqrt{-1}\big(\langle p,q\rangle - \langle q,p\rangle\big) - e^{2\sqrt{-1}t}\big[\|p\|^2 - \|q\|^2 - \sqrt{-1}\big(\langle p,q\rangle + \langle q,p\rangle\big)\big] \\ &\quad + e^{-2\sqrt{-1}t}\big[\|p\|^2 - \|q\|^2 + \sqrt{-1}\big(\langle p,q\rangle + \langle q,p\rangle\big)\big]\big\} \\ &= 0.\end{aligned}$$

By the explicitly formula (7.2) for the geodesic $\sigma(t)$ of F, we see that $\sigma(t)$ is actually a periodic function whose period is 2π. It is clear that σ is a closed geodesic since by (7.2) we have $\sigma(0) = \sigma(2\pi) = p$. Moreover, by Theorem 7.1 we have

$$\|\sigma\|^2 = \|\dot{\sigma}\|^2 = 1, \qquad \big|\langle\sigma,\dot{\sigma}\rangle\big|^2 = 0,$$

from which we get $A = 1$. Thus $F(\sigma,\dot{\sigma}) = \|\dot{\sigma}\|^4 = 1$ and we get

$$L(\sigma) = \int_0^{2\pi} F(\sigma,\dot{\sigma})\,dt = 2\pi. \qquad \square$$

Acknowledgements

This work was done while the author was visiting the University of Notre Dame. The author wishes to thank the Department of Mathematics, especially Professor Jianguo Cao and Professor Mei-Chi Shaw for their help, hospitality and inspiring conversations. The author also wishes to thank the referee for his valuable suggestions and comments which makes this paper greatly improved.

References

[1] M. Abate, G. Patrizio, Finsler Metrics—A Global Approach with Applications to Geometric Function Theory, Lecture Notes in Mathematics, vol. 1591, Springer-Verlag, Berlin/Heidelberg, 1994.
[2] M. Abate, A characterization of the Chern and Berwald connections, Houston J. Math. 22 (4) (1996) 701–717.
[3] N. Aldea, G. Munteanu, On complex Finsler spaces with Randers metric, J. Korean Math. Soc. 46 (5) (2009) 949–966.
[4] T. Aikou, On complex Finsler manifolds, Rep. Fac. Sci., Kagoshima Univ. (Math. Phys. & Chem.) 24 (1991) 9–25.
[5] T. Aikou, Some remarks on locally conformal complex Berwald spaces, Contemporary Mathematics 196 (1996) 109–120.
[6] T. Aikou, Complex manifolds modeled on a complex Minkowski space, J. Math. Kyoto Univ. (JMKYAZ) 35 (1) (1998) 85–103.
[7] D. Bao, S.-S. Chern, Z. Shen, An Introduction to Riemann–Finsler Geometry, Graduate Texts in Mathematics, vol. 200, Springer-Verlag, New York, 2000.
[8] B. Chen, Y. Shen, Kähler Finsler metrics are actually strongly Kähler, Chin. Ann. Math. B 30 (2) (2009) 173–178.
[9] S.S. Chern, Z. Shen, Riemann–Finsler Geometry, World Scientific, 2005.
[10] S. Dragomir, R. Grimaldi, On Rund's connection, Note di Matematica 15 (1) (1995) 85–98.
[11] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1985.
[12] S. Kobayashi, Negative vector bundles and complex Finsler structures, Nagoya Math. J. 57 (1975) 153–166.
[13] S. Kobayashi, Complex Finsler vector bundles, Cont. Math. 196 (1996) 145–153.
[14] M. Matsumoto, Foundations of Finsler Geometry and Special Finsler Spaces, Kaiseisha Press, Saikawa 3-23-2, Otsushi, Shigaken, Japan, 1986.
[15] G. Munteanu, Complex Spaces in Finsler, Lagrange and Hamilton Geometries, Kluwer Academic Publishers, 2004.
[16] G. Munteanu, Totally geodesic holomorphic subspaces, Nonlinear Analysis: Real World Applications 8 (2007) 1132–1143.
[17] H. Rund, The curvature theory of direction-dependent connections on complex manifolds, Tensor N.S. 24 (1972) 189–205.
[18] J.-X. Xiao, R.-M. Yan, Two topics in complex Finsler geometry, Journal of Xiamen University (Natural Science) 45 (5) (2006) 614–616.
[19] R. Yan, Complex Berwald manifolds with vanishing holomorphic sectional curvature, Glasgow Math. J. 50 (2008) 203–208.
[20] C. Zhong, A vanishing theorem on Kaehler–Finsler manifolds, Differential Geometry and Its Applications 27 (2009) 551–565.
[21] C. Zhong, Holomorphic curvature of complex Finsler submanifolds, Science China – Mathematics 53 (2) (2010) 174–261.